THE STOCHASTIC INTEGRAL OF PROCESS MEASURES

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1997

I dedicate this work

To the memory of my mother.
Had she not prevailed upon me
with her superior judgement, I
would have promptly given up
studying Mathematics in the
eleventh grade.
I am proud of everything she
stood for.

To my father with love and respect.
I admire his simplicity and honesty
I am happy that we have begun to
understand each other.

To the memory of 'uncle' Jim Smith
- -whose faith in me never wavered. He
knew me before I knew myself.

and,

To the memory of Veeramma. I have
been deeply touched by her
caring and humility.

ACKNOWLEDGEMENTS

I am infinitely grateful to my advisor Prof. Nicolae Dinculeanu for encouraging me to study under him. Working with him has been a pleasure. I am happy that he had enough faith in my abilities to compel me not to give up studying the Dellacherie-Meyer monograph when I found the going difficult. I have been deeply affected by his humanity. My association with Prof. Dinculeanu has not only made me a better mathematician but also a better human being.

I thank Prof. Murali Rao, Prof. James Brooks, Prof. L. Shen and Prof. M. Nimalendran for taking the time to serve on my supervisory committee.

I have greatly enjoyed the many non-mathematical conversations I have had with Dr. Rao and I hope they continue unabated in the future. I thank Prof. Nimalendran for spending time discussing finance with me. I admit, however, that playing tennis with him was more fun.

I have also benefitted from the mathematical (and philosophical) conversations I have had with Dr. Irene Hueter over the past year. I thank her for her time.

Going back to my undergraduate years, I thank Prof. Richard Bradley, Prof. Gregory Constantine and Prof. Steven Raymar for playing important roles during the various stages of my studies.

I also record my gratitude to the four angels at Indiana University who accompanied me through my first mathematical steps and in the process greatly minimized the turbulence of that period. They are, in alphabetical order, Achab Boudjema, Kai Shun Kwong, Lee Kisuk and Raymond Lai.

Closer home, I thank 'Auntie' Madeleine Smith for having me as her 'rent-a-son.' I have enjoyed her long phone conversations through the years. She has always been there to listen to me. For all this I am grateful.

I am happy to have an understanding brother and caring sister-in-law. They have, together, tried to keep me rooted, like the Banyan trees that grow in Madras.

Finally, I have been fortunate to have friends who have made my life more meaningful.

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of the University of Florida in Partial Fulfillment of the
Requirements for the Degree of Doctor of Philosophy

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December 1997

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In this dissertation we first introduce the notion of a process measure (of which, a martingale measure is a special case). Then, we construct the stochastic integral with respect to p-summable process measures. By doing so we have enlarged the class of integrators from semi-summable processes (as in the work of Brooks and Dinculeanu) to process measures. We then study some of the properties of this stochastic integral. The recurring theme in our construction of the stochastic integral is that the stochastic integral $H \cdot X$ enjoys the same properties as the p-summable process measure X.

The main contribution of this dissertation is that orthogonal martingale measures are 2-summable. This together with the Ito isometry (for stochastic integrals which are process measures) and the representation of the stochastic integral as an orthogonal martingale measure provides a fairly complete L^2 theory of the stochastic integral.

CHAPTER 1
INTRODUCTION

The main results of this dissertation are contained in chapter 4. Section 3 of chapter 4 relies on the work done in chapter 3. Chapter 2 assembles the necessary background required for the remaining chapters. This chapter is separated into 3 sections. In the first section we indicate what we have done–the methods involved and the results obtained. In section two we introduce most of the notation that will be employed in the remainder of this work. Following the notation we list some of the main results of this thesis. Finally in section three we introduce the notion of a bimeasure–as this concept will find occasional use later on (chapter 2 and chapter 4). In a forthcoming article [37], we introduce a theory of integration with respect to operator valued bimeasures in Banach spaces.

1.1 Motivation and Methods

The catalyst which got this thesis started was the paper by Walsh, [47]. In fact Walsh's paper was the seed that spawned a mini cascade of papers [7, 26, 49]. Walsh, in his paper, had introduced the notion of a martingale measure (not in the sense of the term as it is used in the literature in mathematical finance) and then studied a subgroup of these measures, which he termed as worthy martingale measures. Walsh went on to construct a L^2 theory of stochastic integration with respect to the class of worthy martingale measures. His construction of the stochastic integral theory followed, albeit in disguise, the construction of the Ito integral [21, 22, 25, 41, 42].

The notion of a martingale measure gave us the impetus to consider the more general concept of a process measure (see section 1, chapter 2, for the definition).

We then attempted to construct a stochastic integral theory with respect to such measures in a Banach space setting. In this setting the type of integrator we needed to consider for the stochastic integral was the stochastic measure (see section 2, chapter 2), I_X, induced by a process measure, X.

In order to obtain the construction of the stochastic integral we draw upon the theory of integration in Banach spaces [4]. Thus we consider p-summable process measures, X (definition (2.2.1)). That is, we restrict ourselves to those stochastic measures I_X (induced by X) which are σ-additive measures on the underlying σ-algebra and have finite semivariation. This leads us to our work in chapter 3, where we consider the stochastic integral with respect to p-summable process measures. The foundation for this chapter is the paper by Brooks and Dinculeanu [4]. The introduction to chapter 3 contains a detailed description of the work undertaken there. Let us just say that the stochastic integral theory in chapter 3 is new as we have enlarged the class of integrators to process measures. We remark that in section 3 of chapter 4 we obtain a fairly complete L^2 theory of the stochastic integral with respect to an orthogonal martingale measure. We demonstrate here that the familiar Ito isometry in the standard theory carries over to our situation. Finally the recurring theme in chapters 3 and 4 is that the stochastic integral, $H \cdot X$, respects the same properties as the process measure, X. Thus if X is a p-summable process measure (or X is a martingale or X is an orthogonal martingale measure) then $H \cdot X$ correspondingly enjoys these properties.

Having considered a theory of the stochastic integral with respect to p-summable process measures in chapter 3, we show in chapter 4 that there indeed exist process measures that are $p = 2$-summable. Thus the theory of the previous chapter stands on firm ground. In section 2 (of chapter 4) we obtain the fundamental result that orthogonal martingale measures (see definition (2.1.1), part III), with a mild restriction, are 2-summable (relative to $(\mathbb{R}, L^2_E)$, where E is a Hilbert space–theorem (4.2.8)). The proof of this result is an excellent illustration of measure the-

ory at work in stochastic analysis. We consider this result (theorem (4.2.8)) to be the primary contribution of this thesis. In section 3 of chapter 4 we prove that , a mild restriction, the stochastic martingale measure is itself an orthogonal martingale measure. In section 4 of chapter 4 we also prove that a special case of martingale measures with nuclear covariance is 2-summable (relative to $(\mathbb{R}, L^2_E)$)–this is proposition (4.3.1).

1.2 Main Results

In this section we present the main results of this thesis. Our goal here is to indicate, in limited space, what has been accomplished. Thus we limit ourselves to the minimum notation (and discussion) necessary to make the results comprehensible.

NOTATION. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ denote a filtered probability space. The letters E, F and G denote Banach spaces. When we write $c_0 \hookrightarrow G$, we mean that G does not contain an isomorphic copy of c_0. We consider E as continuously embedded in $L(F, G)$ and we write this as $E \subset L(F, G)$. Let $\mathcal{P}$ denote the predictable σ-algebra on $\mathbb{R}_+ \times \Omega$ and let $\mathcal{P}_\infty$ denote the union $\mathcal{P} \cup (\{\{\infty\}\} \times \mathcal{F}_\infty)$. $\mathcal{R}_\mathcal{P}$ will denote the ring of (finite) disjoint unions of rectangles of the form (a) $\{0\} \times A$, $A \in \mathcal{F}_0$ and (b) $(s,t] \times A$, $A \in \mathcal{F}_s$, which generates $\mathcal{P}_\infty$. Let $\mathcal{B}(\mathbb{R})$ denote the Borel σ-algebra on $\mathbb{R}$ and let $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ denote the σ-algebra generated by $\mathcal{P}_\infty$ and $\mathcal{B}(\mathbb{R})$. Let $\mathcal{R}$ denote the ring of (finite) disjoint unions of sets of the form $(s,t] \times A \times (x,y]$ with $A \in \mathcal{F}_s$, $(x,y] \in \mathcal{B}(\mathbb{R})$ which generates $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$. Let X denote a process measure (definition (2.1.1), I) and let M denote a Hilbert valued orthogonal martingale measure (definition (2.1.1), II). Denote by $I_X : \mathcal{R}_\mathcal{P} \times \mathcal{B}(\mathbb{R}) \longrightarrow L^p_E$ the stochastic measure induced by X (see nos. (2.1) and (2.2) in chapter 2). Let $(\tilde{I}_X)_{F,L^p_G} \equiv \tilde{I}_{F,L^p_G}$ denote the semivariation of I_X relative to the spaces (F, L^p_G) (see (4) in section 1 of chapter 3). We say that a process measure, X, is p-summable relative to (F, G) if its corresponding stochastic measure, I_X, satisfies definition (2.2.1). We denote by $\mathcal{F}_{F,G}(X) \equiv \mathcal{F}_{F,L^p_G}(X)$ the space

of all F-valued, $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$-measurable functions H for which $(\widetilde{I}_{F,L^p_G})(H)$ is finite (see definition (2.3.29), proposition (2.3.30) and number (6) in section 1 of chapter 3). The space of functions integrable with respect to I_X is denoted by $L^1_{F,G}(X)$ (recall that the range space of I_X is L^p_E and since $E \subset L(F,G)$ we have $L^p_E \subset L(F, L^p_G)$–see definition (3.2.2)). The stochastic integral $H \cdot X$ is described in definition (3.2.4). The Dólean function $\mu_{\langle M \rangle}$ of $\langle M \rangle$ is defined in proposition (4.3.1). We let $L^1_{\mathbb{R},L^2_E}(\mathcal{B}, M)$ denote the closure of the bounded function in $L_{\mathbb{R},L^2_E}(M)$.

<u>Results</u>

Theorem I. Assume (a) $H \in L^1_{F,G}(X)$ and (b) $\int_D H \; dI_X \in L^p_G$ for every $D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$. Let T be a stopping time. We have

(1) $1_{[[0,T]]}H \in L^1_{F,L^p_G}(X)$ and $H \in L^1_{F,G}(X^T)$.

(2) For any $B \in \mathcal{B}(\mathbb{R})$ and $t \in [0,\infty]$ we have $(H \cdot X)^T_t(B) = ((1_{[[0,T]]}H) \cdot X)_t(B) = (H \cdot X^T)_t(B)$.

Further, if $c_0 \not\hookrightarrow G$ then we have

(3) (a) $(H \cdot X)$, $(H \cdot X)^T$ and $(H \cdot X^T)$ are p-summable relative to $(\mathbb{R}, L^p_G)$.

(b) The process measures $(H \cdot X)^T$ and $H \cdot X^T$ are equal, $(H \cdot X)^T = H \cdot X^T$.

Theorem II. Let E be a Hilbert space and let M be an E-valued orthogonal martingale measure which satisfies the condition:

$$\sup_{B \in \mathcal{B}(\mathbb{R})} E[\langle M(B) \rangle_\infty < \infty.$$

Let $H \in L^1_{\mathbb{R},L^2_E}(\mathcal{B}, M)$ (observe: H is real valued).

Then we have

$$(\widetilde{I}_M)_{\mathbb{R},L^2_E}(H) = || \int H dI_M ||_{L^2_E} = ||H||_{L^2_E(\mu_{\langle M \rangle})} \tag{1}$$

and

$$(\widetilde{I}_M)_{\mathbb{R},L^2_E}(H) = \sqrt{(\widetilde{I}_{\langle M \rangle})_{\mathbb{R},L^1_\mathbb{R}}(H^2)} = \sqrt{(\widetilde{I}_{|M|^2_E})_{\mathbb{R},L^1_\mathbb{R}}(H^2)}. \tag{2}$$

THEOREM III. Let E be a Hilbert space and let M be an E-valued orthogonal martingale measure which satisfies the condition:

$$\sup_{B\in\mathcal{B}(\mathbb{R})} E[\langle M(B)\rangle_\infty] < \infty.$$

Let $H \in L^1_{\mathbb{R},L^2_E}(\mathcal{B}, M)$. Then

(1) $(H \cdot M)$ is a martingale measure which is 2-summable relative to $(\mathbb{R}, L^2_E)$ and

(2) $(H \cdot M)$ is an orthogonal martingale measure.

THEOREM IV. Let $\{M_t(B); (\mathcal{F}_t)_{t\geq 0}, B \in \mathcal{B}(\mathbb{R})\}$ be an orthogonal martingale measure, zero at zero, which satisfies the condition:

$$\sup_{B\in\mathcal{B}(\mathbb{R})} \langle M(B)\rangle_\infty < \infty.$$

Then we have the following.

(i) There is a positive (finite) σ-additive measure $\beta : \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}) \longrightarrow [0,\infty)$, which on a cube in $\mathcal{R}$ of the form $(s,t] \times A \times (x,y]$ with $A \in \mathcal{F}_s$ is given by

$$\beta[(s,t] \times A \times (x,y]] = E[1_A(\langle M(x,y]\rangle_t - \langle M(x,y]\rangle_s)]$$

(ii) The finitely additive stochastic measure $I_M : \mathcal{R} \longrightarrow L^2_E$ is absolutely continuous with respect to β on $\mathcal{R}$ ($\varepsilon - \delta$ form). In fact we have

$$||I_M(C)||_{L^2_E} = \sqrt{\beta(C)} \quad \text{for } C \in \mathcal{R}.$$

(iii) I_M can be extended to a σ-additive measure on $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ with bounded semivariation relative to $(\mathbb{R}, L^2_E)$ given by

$$(\tilde{I}_M)_{\mathbb{R},L^2_E}(D) = ||I_M(D)||_{L^2_E} \quad \text{for } D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}).$$

PROPOSITION V. Let M be an E-valued orthogonal martingale measure, zero at zero, which satisfies the condition:

$$\sup_{B\in\mathcal{B}(\mathbb{R})} E[\langle M(B)\rangle_\infty] < \infty.$$

Then $\langle M\rangle$ is a process measure that is summable relative to $(\mathbb{R}, L^1_{\mathbb{R}})$. The semivariation of $I_{\langle M\rangle}$, $(\tilde{I}_{\langle M\rangle})_{\mathbb{R},L^1_{\mathbb{R}}}$, can be computed as follows.

$$(\tilde{I}_{\langle M\rangle})_{\mathbb{R},L^1_{\mathbb{R}}}(D) = ||I_{\langle M\rangle}(D)||_{L^1_{\mathbb{R}}},\ D\in\mathcal{P}_\infty\otimes\mathcal{B}(\mathbb{R}).$$

1.3 Bimeasures

In this section we briefly discuss the notion of a bimeasure. This concept is used sparingly in section 2 of chapter 2. In theorem (4.2.8) we make use of lemma (1.3.3) below. We direct the reader interested in bimeasures (and their integral extensions) to the following references: [6, 14, 15, 34, 35, 36, 37, 39, 40, 50, 51, 52].

DEFINITION (1.3.1).

(a) We say that a set function $\beta : \Sigma_1 \times \Sigma_2 \longrightarrow D \subset L^{(2)}(X_1 \times X_2, Y)$ is a vector bimeasure if it is separately σ-additive, that is if

  (i) $\beta(E,\cdot) : \Sigma_2 \longrightarrow D$ is a countably additive vector measure for each $E\in\Sigma_1$ and

  (ii) $B(\cdot, F) : \Sigma_1 \longrightarrow D$ is a countably additive vector measure for each $F\in\Sigma_2$.

(b) If the countable additivities in (i) and (ii) are uniform we say that β is a uniform vector bimeasure.

REMARK (1.3.2).

(a) It turns out that each scalar bimeasure (that is, when Y is replaced by $\mathbb{C}$) is uniform (52, Theorem 4.4)

(b) There are bimeasures that are not uniform. An example can be found in Dobrakov [14].

(c) We bring to the reader's attention that bimeasure theory, even in the scalar case, is not just the theory of measures. Thus it differs from the theory of product measures. It is known (see Morse and Transue [36], page 480, where references are given) that a bimeasure $\beta : \Sigma_1 \times \Sigma_2 \longrightarrow \mathbb{R}$ does not admit a decomposition of the form

$$\beta = \alpha - \gamma$$

where α and γ are two positive bimeasures on $\Sigma_1 \times \Sigma_2$.

The next lemma says that a bimeasure is, infact, a finitely additive function on $\Sigma_1 \times \Sigma_2$. A proof of this result may be found in Muthiah [37].

LEMMA (1.3.3). Let $\beta : \Sigma_1 \times \Sigma_2 \longrightarrow D \subset L^{(2)}(X_1 \times X_2, Y)$ be a bimeasure. Extend β to $\mathcal{R}(\Sigma_1 \times \Sigma_2)$, the ring of finite disjoint unions of elements in $\Sigma_1 \times \Sigma_2$. That is, if $(A_i, B_i)_{i=1}^n \equiv (A_i \times B_i)_{i=1}^n$ are disjoint elements of $\Sigma_1 \times \Sigma_2$, then set

$$\beta\left(\bigcup_{i=1}^n A_i \times B\right) = \sum_{i=1}^n \beta(A_i \times B_i).$$

The set function β extended in this manner is well defined.

REMARK (1.3.4). Lemma (1.3.3) is still time if B is only separately finitely additive in each component.

CHAPTER 2
PROCESS MEASURES, SUMMABILITY AND INTEGRATION IN BANACH SPACES

The purpose of this chapter is to assemble the necessary background for this dissertation. This chapter is divided into three sections. In the first section we introduce the concept of a process measure and some of its specializations. The definition of a process measure is a natural outgrowth of the already existing notion of a martingale measure (not in the sense of the term as it is used in mathematical finance). Process measures will play a fundamental role in the theory of stochastic integration that we consider in the next chapter. In fact, there, we consider stochastic integrals where the integrator is a stochastic measure, I_X, induced by a process measure, X. In section two we discuss some properties of I_X (such as summability) which will be needed later. Finally in the third section we summarize the general integration theory in Banach spaces (this is taken from the appendix of Brooks and Dinculeanu [4]). This theory serves as the foundation for the stochastic integration theory that we introduce in the next chapter.

We now introduce the notation that will be employed in this chapter (and the following ones). Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ denote a filtered probability space satisfying the 'usual conditions'. By 'usual conditions' we, of course, mean that (1) $\mathcal{F}_0$ is complete and (2) $\mathcal{F}_t$ is right continuous for each $t \geq 0$, that is, $\mathcal{F}_{t_+} = \mathcal{F}_t$. We specialize the σ-algebras Σ_1 and Σ_2 in section 3 of chapter 1 to $\Sigma_1 = \mathcal{P}_\infty$ and $\Sigma_2 = \mathcal{L}$, where $\mathcal{P}_\infty$ denotes the predictable σ-algebra on $\mathbb{R}_+ \times \Omega$ (see [9]) and $\mathcal{L}$ denotes the σ-algebra on some underlying Lusin space. In fact in the following chapters we shall replace $\mathcal{L}$ by $\mathcal{B}(\mathbb{R})$ (the Borel σ-algebra on $\mathbb{R}$) as there is no loss of generality. Let

E, F, G denote Banach spaces. The notation $L(F, G)$ denotes the Banach space of bounded linear operators from F to G. When we write $E \subset L(F, G)$ we mean that E is continuously embedded in $L(F, G)$.

2.1 Process Measures

DEFINITION (2.1.1).

(I) Let $X : \mathbb{R}_+ \times \Omega \times \mathcal{L} \longrightarrow E \subset L(F, G)$ be cadlag (that is, right continuous with left limits) such that $X_t(B) \in L^p_E(P) \equiv L^p_E$, where $1 \leq p < \infty$. We say that $\{X_t(B),\ t \geq 0,\ B \in \mathcal{L}\}$ is a Process Measure iff:

(a) $\{X_t(B), t \geq 0\}$ is $\mathcal{F}_t$-adapted, $\forall\ B \in \mathcal{L}$.

(b) For each $t \geq 0$, $X_t(\cdot)$ is a L^p_E-valued σ-additive measure.

(II) We say that $\{X_t(B), t \geq 0, B \in \mathcal{L}\}$ is a Martingale Measure if it satisfies condition (b) in (I) and it satisfies (b′) below

(b′) $\{X_t(B), t \geq 0\}$ is a $\mathcal{F}_t$-martingale, for each $B \in \mathcal{L}$.

(III) Let $p = 2$ and E a Hilbert space. An Orthogonal Martingale Measure is a martingale measure that satisfies condition (d) below

(d) $X(A)$ and $X(B)$ are orthogonal martingales for $A, B \in \mathcal{L}$ and $A \cap B = \emptyset$. ('orthogonal' here means that the sharp bracket of $X(A)$ and $X(B)$, that is $\langle X(A), X(B)\rangle$, is zero.)

(IV) A martingale measure has Nuclear Covariance if there exists a finite measure μ on $(L, \mathcal{L})$ and a complete orthonormal system (CONS) of real valued step functions $(\phi_k)_{k=1}^{\infty}$ in $L^2(L, \mathcal{L}, \mu)$ such that

(i) $\mu(A) = 0 \Longrightarrow E[(X_t(A))^2] = 0$, $\forall A \in \mathcal{L}$, $\forall\ t \in \mathbb{R}_+$ and

(ii) $\sum_{k=1}^{\infty} E[(X_t(\phi_k))^2] < \infty$, $\forall\ t \in \mathbb{R}_+$.

Note: For (ii) above we define $X_t(\phi_k)$ as follows. If $\phi_k = \sum_{i=1}^{i_k} a_i I_{A_i}$ then define $X_t(\phi_k)$ by

$$X_t(\phi_k) = \sum_{i=1}^{i_k} a_i X_t(A_i).$$

In chapter 4 we prove that orthogonal martingale measures that satisfy a certain condition are 2-summable (see Section 2 of this chapter for definitions). We also prove there that a special case of martingale measures with nuclear covariance is 2-summable.

2.2 The Stochastic Measure I_X

Let X be a process measure. Denote by $\mathcal{R}_\mathcal{P}$ the ring of finite disjoint unions of predictable rectangles of the form $(s,t] \times A$ with $0 \leq s < t < \infty$, $A \in \mathcal{F}_s$ and $\{0\} \times B = [0_B]$ with $B \in \mathcal{F}_0$. Note that $\mathcal{R}_\mathcal{P}$ generates $\mathcal{P}$. Define a finitely additive stochastic measure $I_X : \mathcal{R}_\mathcal{P} \times \mathcal{L} \longrightarrow L_E^p$ first for cubes by

$$I_X([0_A] \times B) = 1_A X_0(B), \; A \in \mathcal{F}_0, \; B \in \mathcal{L} \tag{2.1}$$

and

$$I_X(((s,t] \times A) \times B) = 1_A[X_t(B) - X_s(B)], \; A \in \mathcal{F}_s, \; B \in \mathcal{L}. \tag{2.2}$$

and then extend I_X in an additive fashion to $\mathcal{R}(\mathcal{R}_\mathcal{P} \times \mathcal{L})$, the ring of disjoint unions of rectangles from $\mathcal{R}_\mathcal{P} \times \mathcal{L}$. Observe that

$$I_X(([0,t] \times \Omega) \times B) = X_t(B), \quad for \;\; t \geq 0; \tag{2.3}$$

for the left hand side of (2.3) can be written as

$$I_X[(\{0\} \times \Omega) \times B)\dot{\bigcup}((0,t] \times \Omega) \times B] = 1_\Omega X_0(B) + 1_\Omega[X_t(B) - X_0(B)] = X_t(B).$$

Following the usual conventions we denote by $X_{\infty_-}(\cdot)$ the limit of $X(\cdot)$ at ∞ if it exists [9]. If X is a summable process measure (see theorem (2.5.5), part (a)) then the left limit of $X(\cdot)$ exists at ∞ in L_E^p.

Semivariation of I_X.

The reader is directed to section 3 for the notion of semivariation. We note here that the inclusion $E \subset L(F,G)$ implies that $L^p_E \subset L(F, L^p_G)$; hence we may compute the semivariation of I_X relative to the pair (F, L^p_G). We denote this semivariation by $\tilde{I}_{F,G}$ ($\equiv \tilde{I}_{F,L^p_G}$) and it is defined by

$$\tilde{I}_{F,G}(C) = \sup \| \sum I_X(C_i)x_i \|_{L^p_G} \tag{2.4}$$

where $C \in \mathcal{R}(\mathcal{R}_{\mathcal{P}} \times \mathcal{L})$ and the sup in (2.4) is taken over all finite families of elements $(x_i) \in F$ with $|x_i| \leq 1$ and disjoint sets (C_i) from $\mathcal{R}(\mathcal{R}_P \times \mathcal{L})$ contained in C. If I_X can be extended to $\mathcal{P} \otimes \mathcal{L}$, then $\tilde{I}_{F,G}$ is defined similarly, except now on sets in $\mathcal{P} \otimes \mathcal{L}$. We say that I_X has finite semivariation relative to (F, L^p_G) if $\tilde{I}_{F,G}(C) < \infty$, for every C in $\mathcal{R}(\mathcal{R}_{\mathcal{P}} \times \mathcal{L})$ (or $\mathcal{P} \otimes \mathcal{L}$ as the case may be).

When Can We Extend I_X to $R(P \times \mathcal{L})$?

For ease of discussion in this section denote I_X by the symbol β. That is, let $\beta : \mathcal{R}_{\mathcal{P}} \times \mathcal{L} \longrightarrow L^p_E$ be given by $\beta((s,t] \times A, B) = I_X(((s,t] \times A) \times B)$. From (2.1) and (2.2) we realize that for each $(s,t] \times A$ fixed, $\beta((s,t] \times A, \cdot)$ is a L^p_E valued σ-additive measure (remember that X is a process measure) and for each B fixed, $\beta(\cdot, B)$ is finitely additive (as an element of E). An immediate question one may ask now is as follows: under what conditions can we extend $\beta(\cdot, B)$ to a σ-additive measure on $\mathcal{P}_\infty$? The extension theorem in Brooks and Dinculeanu [5] (Theorem 28, page 345) provides us with a condition that answers this question affirmatively. The condition is given in (2.5) below.

Condition. Assume (i) E does not contain a copy of c_0 and (ii) $\beta(\cdot, B)$ is bounded on $\mathcal{R}_{\mathcal{P}}$. (2.5)

If we now allow the condition in (2.5) to be in force, the next natural question to ask is whether $\beta : \mathcal{P}_\infty \times \mathcal{L} \longrightarrow L^p_E$ is a bimeasure. Recall that $\beta(A, \cdot)$ is σ-additive in $\mathcal{L}$ for each $A \in \mathcal{R}$. What we would like to know is whether $\beta(A, \cdot)$ is σ-additive in $\mathcal{L}$ for each $A \in \mathcal{P}_\infty$. The answer to this question is indeed yes. Let us provide a proof.

Denote by $\mathcal{M}$ the set

$$\mathcal{M} = \{A \in \mathcal{P}_\infty : \beta(A, \cdot) \text{ is a } \sigma\text{-additive measure on } \mathcal{L}\}.$$

We know that $\mathcal{R}_\mathcal{P} \subset \mathcal{M}$. We show that $\mathcal{M}$ is a monotone class. Hence let (A_n) be a sequence in $\mathcal{M}$ and suppose that $(A_n) \uparrow (\downarrow) A$ and prove that $A \in \mathcal{M}$. For each $B \in \mathcal{L}$, we have

$$\beta(A_n, B) \longrightarrow \beta(A, B). \tag{2.6}$$

Applying the generalized Nikodym Theorem ([18], page 321) to the family of measures $\{\beta(A_n, \cdot)\}_{n=1}^{\infty}$ and using (2.6) we obtain that $\beta(A, \cdot)$ is a σ-additive measure on $\mathcal{L}$. It follows that $\mathcal{M}$ is a monotone class and hence $\mathcal{M} = \mathcal{P}$. Thus β is a bimeasure.

Summable Processes

The above line of thought leads us to ask when I_X can be extended to a L^p_E valued σ-additive measure on $\mathcal{P}_\infty \otimes \mathcal{L}$. The next definition formalizes this notion.

Definition (2.2.1). We say that a process measure X is p-summable relative to (F, G) if its corresponding stochastic measure, which is additive on $\mathcal{R}(\mathcal{R}_\mathcal{P} \times \mathcal{L})$, can be extended to a σ-additive L^p_E valued measure on the σ-algebra $\mathcal{P}_\infty \otimes \mathcal{L}$ with finite semivariation (Section 3) relative to (F, L^p_G).

Remark (2.2.2). We do not have a general criterion (such as (2.5)) to determine if X is p-summable. However, in chapter 4 we prove that X is p-summable under each of the following conditions:

(1) X is an orthogonal martingale measure.

(2) X is a special case of a martingale measure with nuclear covariance.

Remark (2.2.3). The notion of summability of a process measure is important because it allows us to construct the stochastic integral with respect to X (see chapter 3).

Remark (2.2.4).

(a) When $p = 1$ we simply say that X is summable relative to (F, G). Suppose we consider E as $E = L(\mathbb{R}, E)$ and if X is p-summable relative to $(\mathbb{R}, E)$ then we just say X is p-summable.

(b) If $1 \leq p' \leq p < \infty$ and if X is p-summable relative to (F, G), then X is p'-summable relative to (F, G). This is seen immediately if one recalls that $L^p_G \subset L^{p'}_G$ on a finite measure space.

(c) If X is p-summable relative to (F, G) then for any $t \geq 0$ we have $X_{t-}(\cdot) \in L^p_E$ and $I_X(([0,t)\times\Omega)\times B) = X_{t-}(B)$. For let $t_n \uparrow t$, then $I_X(([0,t_n]\times\Omega)\times B) = X_{t_n}(B)$ (by (2.2)) and $X_{t_n}(B) \longrightarrow X_{t-}(B)$ as $X_t(\cdot)$ is cadlag. By the σ-additivity of I_X on $\mathcal{P}_\infty \otimes \mathcal{L}$ we have $X_{t_n}(B) = I_X(([0,t_n] \times \Omega) \times B) \longrightarrow I_X(([0,t) \times \Omega) \times B)$ in L^p_E. Hence $I_X(([0,t) \times \Omega) \times B) = X_{t-}(B)$ P.a.e. and $X_{t-}(B) \in L^p_E$.

Computing I_X on Stochastic Intervals.

The predictable σ-algebra $\mathcal{P}$ is generated by stochastic intervals of the form [9]

$$((S,T]] = \{(t,w) \in \mathbb{R}_+ \times \Omega : S(w) < t \leq T(w)\}, \tag{2.7}$$

where S, T are stopping times with $S \leq T$.

The other stochastic intervals in $\mathcal{P}_\infty$ are similarly defined (((S,T]],[[S]], ([S,T)) etc.). We shall at this point extend the σ-algebra $\mathcal{P}$ to $\mathcal{P}_\infty = \mathcal{P} \cup (\{\{\infty\}\} \times \mathcal{F}_\infty)$ where $\mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t$. Then, if X is p-summable, we extend it to $\mathcal{P}_\infty \otimes \mathcal{L}$ by setting $I_X((\{\{\infty\}\} \times A) \times B) = 0$, where $A \in \mathcal{F}_\infty$.

We now present a theorem which extends I_X to stochastic intervals. The reader will note that in the statement of this theorem the set $B \in \mathcal{L}$ is fixed. Hence this theorem is really nothing but theorem 2.2 [4]. This theorem is, however, about process measures and hence is an extension of the theorem in Brooks and Dinculeanu [4]. It will turn out to be extremely useful in the next chapter.

THEOREM (2.2.5). Let X be a process measure which is p-summable relative to (F, G) and let $B \in \mathcal{L}$.

(a) The left limit at ∞ of $X(B)$, which we denote by $X_\infty(B)$, exists in L^p_E. That is, we have $\lim_{t\to\infty} X_t(B) = X_\infty(B)$ in L^p_E. For $A \in \mathcal{F}_t$, we have

$$I_X[((t,\infty) \times A) \times B] = 1_A[X_\infty(B) - X_t(B)].$$

If $X(B)$ has a pointwise left limit then $X_\infty(B) = X_{\infty-}(B)$ a.s.

For parts (b), (c) and (d) consider $X_{\infty-}(B) = X_\infty(B)$ a.s.

(b) For any stopping time T, we have $X_T(B) \in L^p_E$ and $I_X([[0,T]] \times B) = X_T(B)$.

(c) In the case that T is a predictable stopping time we have $X_{T-}(B) \in L^p_E$ and $I_X([[0,T)) \times B) = X_{T-}(B)$ and $I_X([[T]] \times B) = \Delta X_T(B)$.

(d) Suppose S and T are stopping times with $S \leq T$. Then $I_X\{((S,T]] \times B\} = X_T(B) - X_S(B)$. We also have the following three cases:

(i) $I_X([[S,T]] \times B) = X_T(B) - X_{S-}(B)$ if S is predictable.

(ii) $I_X(((S,T)) \times B) = X_{T-}(B) - X_{S-}(B)$ if T is predictable and

(iii) $I_X([[S,T)) \times B) = X_{T-}(B) - X_{S-}(B)$ if both S and T are predictable.

2.3 Integration in Banach Spaces

This section is mainly a summary of results from the theory of general bilinear vector integration with respect to a Banach valued measure as developed in the appendix of Brooks and Dinculeanu [4]. For more details the reader may consult Diestel and Uhl [11] and Brooks and Dinculeanu [3]. We also provide a brief outline of this theory. We apply this theory in chapter 3 to construct a stochastic integral with respect to p-summable process measures.

NOTATION (2.3.0). (0) For the rest of this section the letter m will denote either a finitely or countably additive measure on a ring $\mathcal{R}$ or a σ-algebra Σ, which is generated by $\mathcal{R}$. (1) The letters E, F, G for the remainder of this paper will represent

Banach spaces. The norm on these spaces will be denoted by $|\cdot|$ (or $|\cdot|_E, |\cdot|_F$ and $|,|_G$ if we need to be specific). (2) The dual of a Banach space X will be denoted by X^* and the unit ball of X by X_1. (3) We shall write $L(F, G)$ for the space of bounded linear operators from F to G. (4) The inclusion $E \subset L(F, G)$ will be taken to mean that E is continuously embedded in $L(F, G)$. Two examples of this are $E = L(\mathbb{R}, E)$ and $E \subset L(E^*, \mathbb{R}) = E^{**}$. (5) We say that a subset $Z \subset E^*$ is norming for E if for every $x \in E$, $|x|$ can be written as

$$|x| = \sup\{|\langle x, z\rangle| : z \in Z_1\}.$$

Observe that E^* is norming for E and $E \subset E^{**}$ is norming for E^* (by an application of a corollary to the Hahn-Banach Theorem). We record now an important example of a norming space for later use. Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $1 \leq p \leq \infty$ and let q be conjugate to p, that is $\frac{1}{p} + \frac{1}{q} = 1$. Then $L^q_{E^*} \subset (L^p_E)^*$ and $L^q_{E^*}$ is norming for L^p_E (see Dinculeanu [12] , page 232). Further if $\mathcal{R}$ is a ring generating $\mathcal{F}$ (the σ-algebra for the underlying space E), then the E^* valued simple functions over $\mathcal{R}$ form a norming space for L^p_E.

DEFINITION (2.3.1). We call a family $(m_\alpha)_{\alpha \in I}$ of E-valued measures on the ring $\mathcal{R}$ uniformly σ-additive if for every decreasing sequence $A_n \downarrow \emptyset$ of sets from $\mathcal{R}$ we have

$$\lim_{n \longrightarrow \infty} m_\alpha(A_n) = 0$$

uniformly with respect to α.

REMARK (2.3.2). A finitely additive measure $m : \mathcal{R} \longrightarrow E$ is σ-additive iff the family $\{x^*m; x^* \in E_1^*\}$ of scalar measures is uniformly σ-additive. The measure $x^*m : \mathcal{R} \longrightarrow \mathbb{R}$ is defined by

$$(x^*m)(A) = \langle m(A), x^*\rangle, \quad \text{for} \quad A \in \mathcal{R}.$$

The Variation of a Measure.

DEFINITION (2.3.3.). Let $m : \mathcal{R} \longrightarrow E$ be a finitely additive measure. The <u>variation</u> of m is the set function $|m| : \mathcal{R} \longrightarrow \mathbb{R}_+$, defined for each $A \in \mathcal{R}$ by

$$|m|(A) = \sup \Sigma |m(A_i)|$$

where the sup is taken over all finite families (A_i) of disjoint subset from $\mathcal{R}$ with $\cup A_i = A$.

REMARK (2.3.4). m is finitely (countably) additive iff $|m|$ is finitely (countably) additive (see Dinculeanu [12], Proposition 10, page 41).

DEFINITION (2.3.5). A measure $m : \mathcal{R} \longrightarrow E$ has <u>finite variation</u> if

$$|m|(A) < \infty \quad \forall A \in \mathcal{R}.$$

The measure m has <u>bounded variation</u> if $\sup\{|m|(A);\, A \in \mathcal{R}\} < \infty$.

REMARK (2.3.6). If $m : \mathcal{R} \longrightarrow \mathbb{R}$ is a bounded measure then m has bounded variation. This follows since for a real valued measure we have

$$|m|(A) \leq 2 \sup_{B \subseteq A} |m(B)|$$

(see, for example, Proposition 7, page 39 in Dinculeanu [12]).

Finally let $m : \Sigma \longrightarrow E$, where Σ is a σ-algebra, be σ-additive. Then a set or a function is called m-negligible if it has the same property with respect to $|m|$. For any Banach space F we shall denote the F-valued integrable functions with respect to m by $L^1_F(m) \equiv L^1_F(|m|)$ and we endow this space with the semi-norm $||f||_1 = \int |f| d|m|$. If $E \subset L(F, G)$, then for any simple function $f \in L^1_F(m)$, we define $\int f \; dm \in G$ in the standard way. For a general $f \in L^1_f(m)$, the integral is obtained as the limit of the integrals of step functions.

We now present an important extension theorem which will be necessary for the construction of the stochastic integral.

THEOREM (2.3.7). Assume that E does not contain (an isomorphic) copy of c_0 and let $Z \subset E^*$ be a norming space for E. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $1 \leq p < \infty$,

and let $m : \mathcal{R} \longrightarrow L^p_E(\mu)$ be a finitely additive measure. For each $z \in Z$ define the measure $zm : \mathcal{R} \longrightarrow L^p(\mu)$ by $(zm)(A) = \langle m(A), z\rangle \equiv \int \langle m(A)(w), z(w)\rangle \, d\mu$, for $A \in \mathcal{R}$. If m is bounded on $\mathcal{R}$ and if for each $z \in Z$, the measure zm is σ-additive on $\mathcal{R}$, then m can be uniquely extended to a σ-additive measure $m_1 : \Sigma \longrightarrow L^p_E(\mu)$, where Σ is the σ-algebra generated by the ring $\mathcal{R}$.

Semivariation of a Measure

DEFINITION (2.3.8). (1) The Semivariation relative to the pair (F, G) of a finitely additive measure $m : \mathcal{R} \longrightarrow E \subset L(F, G)$ on any set $A \in R$ is denoted by $\widetilde{m}_{F,G}(A)$ and it is defined as

$$\widetilde{m}_{F,G}(A) = \sup \left| \sum_{i \in I} m(A_i) x_i \right|$$

where the supremum is taken over all finite families $(A_i)_{i\in I}$ of disjoint sets from $\mathcal{R}$, with union A, and all finite families $(x_i)_{i\in I}$ of elements from F_1. (2) We say that m has finite (bounded) semivariation relative to (F, G) if $\widetilde{m}_{F,G}(A) < \infty$ for every $A \in \mathcal{R}$ $(\sup\{\widetilde{m}_{F,G}(A) : A \in \mathcal{R}\} < \infty)$.

REMARK (2.3.9). Observe that we may write $\widetilde{m}_{F,G}(A)$ above as

$$\widetilde{m}_{F,G}(A) = \sup \left| \int h \, dm \right|,$$

where the sup is taken over all F-valued $\mathcal{R}$-measurable simple functions h such that $|h|_F \leq 1_A$, where $\int h \, dm$ is defined as usual.

LEMMA (2.3.10). Let $m : \mathcal{R} \longrightarrow E \subset L(F, G)$ be finitely additive. For $z \in E^*$, define the set function $m_z : \mathcal{R} \longrightarrow \mathbb{R}$ by $m_z(A) = \langle m(A), z\rangle$. Then m_z is a finitely additive measure. If m is countably additive, then so is m_z.

The proof of this lemma is straightforward and may be found in Dinculeanu [12], page 39.

The next lemma (already observed in remark (2.3.6)) is a standard result. We state it explicitly for future reference.

LEMMA (2.3.11). Let $m : \mathcal{R} \longrightarrow \mathbb{R}$ be a σ-additive measure. Then for any set

$A \in \mathcal{R}$, $|m|(A)$, where $|m|$ denotes the variation of m, has the following bound:

$$|m|(A) \leq 2 \sup_{B \subset A \in \mathcal{R}} |m(B)|.$$

A proof of this lemma can be found on page 39 of Dinculeanu [12].

A useful fact concerning the semivariation of a measure that will come in handy is the next lemma.

LEMMA (2.3.12). The semivariation of a set function $m : \mathcal{R} \longrightarrow E \subset L(F, \mathbb{C}) = F^*$ relative to $(F, \mathbb{C})$ is equal to the variation of m.

A proof of this lemma can be found on page 54 of Dinculeanu [12].

The next proposition says that the semivariation relative to (F, G), say, of a measure m can also be computed in terms of the variation of a measure with finite variation.

PROPOSITION (2.3.13). Let $m : \mathcal{R} \longrightarrow E \subset L(F, G)$ be a finitely additive measure. For each $z \in G^*$ denote by $m_z : \mathcal{R} \longrightarrow L(F, \mathbb{C}) = F^*$ the set function defined by the equality $m_z(A)x = \langle m(A)x, z \rangle$, for $A \in \mathcal{R}$ and $x \in F$. Let $Z \subset G^*$ be norming for G and assume that $m(\emptyset) = 0$. Then for every $A \in \mathcal{R}$ we have

$$\widetilde{m}_{F,G}(A) = \sup_{|z| \leq 1, z \in Z} |m_z|(A).$$

We remark that if m is finitely additive (respectively, countably additive) then m_z has the corresponding property. A proof for the above proposition may be found on page 55 of Dinculeanu [12]. The next proposition is an application of the earlier results.

PROPOSITION (2.3.14). Let $m : \mathcal{R} \longrightarrow E = L(\mathbb{R}, E)$ be finitely additive. Then m has bounded semivariation relative to $(\mathbb{R}, E)$ iff m is bounded on $\mathcal{R}$.

Proof. Let $Z \subset E^*$ be norming for E and let $A \in \mathcal{R}$. Assume that m is bounded on $\mathcal{R}$. We then have $\sup\{|m(B)|; B \subset A, B \in R\} < \infty$. Computing $\widetilde{m}_{\mathbb{R},E}(A)$ we have

$$\begin{aligned} \widetilde{m}_{\mathbb{R},E}(A) &= \sup_{z \in E_1^*} |m_z|(A) \leq 2 \sup_{\substack{B \subset A, B \in \mathcal{R} \\ z \in E_1^*}} |m_z(B)| \\ &= 2 \sup_{\substack{B \subset A, B \in R \\ z \in E_1^*}} |\langle m(B), z \rangle| \leq 2 \sup_{\substack{B \subset A \\ B \in \mathcal{R}}} |m(B)| < \infty \end{aligned} \tag{2.8}$$

The first equality in (2.8) is by proposition(2.3.13) and the first inequality in (2.8) is by lemma (2.3.11) as m_z is a scalar measure. Thus m is of bounded semivariation relative to $(\mathbb{R}, E)$ on $\mathcal{R}$. Conversely assume m is of bounded semivariation relative to $(\mathbb{R}, E)$. Then $\sup\{\widetilde{m}_{\mathbb{R},E}(A); A \in \mathcal{R}\} < \infty$. That is,

$$\sup_{A\in\mathcal{R}} \sup_{(A_i),(x_i)} \left\{ |\Sigma m(A_i)x_i|; \begin{array}{l} \text{the families } (A_i) \text{ and } (x_i) \text{ are} \\ \text{as in definition (2.3.8) with } |x_i| \leq 1 \end{array} \right\} < \infty. \qquad (2.9)$$

In particular take all the x_i to be 1. The statement in (2.9) then becomes $\sup_{A\in\mathcal{R}} |m(A)| < \infty$. That is, m bounded on $\mathcal{R}$. ■

The next proposition compares the semivariation of the finitely additive measure $m : \mathcal{R} \longrightarrow E \subset L(F,G)$ computed relative to various pairs of spaces.

PROPOSITION (2.3.15). Let $m : \mathcal{R} \longrightarrow E \subset L(F,G)$ be finitely additive (the inclusion $E \subset L(F,G)$ means that E is isometrically embedded in $L(F,G)$). We have

$$\widetilde{m}_{\mathbb{R},E} \leq \widetilde{m}_{F,G} \leq \widetilde{m}_{E^*,\mathbb{R}}.$$

For a proof see Dinculeanu [12], page 51. Let us return to proposition (2.3.13) for a moment. We know from there that if $\widetilde{m}_{F,G}$ is bounded, then for each $z \in Z$ $|m_z|$ is bounded (infact the family $\{|m_z|\}_{z\in Z}$ is uniformly bounded). In the case that Z is a closed subspace of G^* the converse is also true (5, page 326). We state this converse.

PROPOSITION (2.3.16). Let $m : \mathcal{R} \longrightarrow E \subset L(F,G)$ be finitely additive. Assume Z is a closed subspace of G^*, norming for G. If the variation $|m_z|$ is bounded on $\mathcal{R}$ for each $z \in Z$, then $\widetilde{m}_{F,G}$ is bounded on $\mathcal{R}$.

The next proposition is essential in establishing the fact that an orthogonal martingale measure is 2-summable, which is done in chapter 4.

PROPOSITION (2.3.17). Let $m : \mathcal{R} \longrightarrow E \subset L(F,G)$ be a finitely additive measure with bounded semivariation $\widetilde{m}'_{F,G}$. If m has a σ-additive extension m' to Σ, then m' has bounded semivariation $\widetilde{m}_{F,G}$ on Σ and $\widetilde{m}'_{F,G}$ is the extension of $\widetilde{m}_{F,G}$.

For a proof see page 326 in Brooks and Dinculeanu [5].

NOTATION (2.3.18). Let $m : \Sigma \longrightarrow E \subset L(F,G)$ be a finitely additive measure. Denote by the symbol $m_{F,G}$ the set of positive measures given by

$$m_{F,G} = \{|m_z| : z \in Z_1\}.$$

REMARK (2.3.19). Observe that $m_{F,G}$ is uniformly σ-additive iff $\widetilde{m}_{F,G}(A_n) \longrightarrow 0$ whenever $A_n \downarrow \emptyset$ (apply proposition 2.3.13).

An important question that one may ask is: Under what conditions is the set $m_{F,G}$ uniformly σ-additive? The next proposition answers this question [14].

PROPOSITION (2.3.20). Let $m : \Sigma \longrightarrow L(F,G)$ be σ-additive and suppose that $c_0 \not\subset G$. Then $m_{F,G}$ is uniformly σ-additive on Σ.

REMARK (2.3.21). Infact, in the previous proposition Σ may be replaced by $\mathcal{R}$ (5, page 327).

Measurable and Negligible Notions with Respect to $m_{F,G}$

CAUTION (2.3.22). Henceforth we will assume that $m : \Sigma \longrightarrow E \subset L(F,G)$ is finitely additive and has bounded semivariation $\widetilde{m}_{F,G}$. The letter $Z \subset G^*$ will denote a norming space for G. We will also assume that the elements $|m_z|$ of $m_{F,G}$ are σ-additive measures.

Let S denote the underlying space for Σ. We will consider F-valued functions on S, $f : S \longrightarrow F$.

DEFINITION (2.3.23). A set $Q \subset S$ is said to be m-negligible if there exists a set $A \in \Sigma$ with $Q \subset A$, such that $m(B) = 0$ for every $B \subset A$, $B \in \Sigma$.

From the definition above it follows that a set $A \in \Sigma$ is m-negligible iff $\widetilde{m}_{F,G}(A) = 0$.

DEFINITION (2.3.24). Let D be any Banach space. A function $f : S \longrightarrow D$ is said to be m negligible (i.e., $f = 0$ m-a.e.) if it vanishes outside an m-negligible set.

DEFINITION (2.3.25). Let Q be any subset of S. We say that Q is $m_{F,G}$ negligible if for each $z \in Z$, Q is contained in an $|m_z|$-negligible set.

DEFINITION (2.3.26). (1) We call a function $f : S \longrightarrow D$ $m_{F,G}$-measurable if it is m_z-measurable for every $z \in Z$. (2) We shall say that $f : S \longrightarrow D$ is m-measurable

if it is the m- a.e. limit of a sequence of D-valued Σ-measurable simple functions.

REMARK (2.3.27). A little computation will show that if f is m-measurable then f is $m_{F,G}$-measurable. The next proposition says that the converse is true provided $m_{F,G}$ is uniformly σ-additive (5, page 328).

PROPOSITION (2.3.28). Suppose $m_{F,G}$ is uniformly σ-additive. Then a function $f : S \longrightarrow D$ is m-measurable iff it is $m_{F,G}$-measurable.

The Semivariation of a Function

DEFINITION (2.3.29). Let $f : S \longrightarrow D$ (or $\overline{\mathbb{R}}$) be a $m_{F,G}$-measurable function. We define $\widetilde{m}_{F,G}(f)$ as

$$\widetilde{m}_{F,G}(f) = \widetilde{m}_{F,G}(|f|) = \sup \left\{ \left| \int s \, dm \right| \right\},$$

where the supremum is extended over all F-valued, Σ-measurable simple functions s such that $|s| \leq |f|$ on S.

Observe that for $A \in \Sigma$ we have $\widetilde{m}_{F,G}(A) = \widetilde{m}_{F,G}(1_A)$. The next proposition is similar to proposition (2.3.13) (see Brooks and Dinculeanu [5], page 329, for a proof).

PROPOSITION (2.3.30). Let $f : S \longrightarrow D$ be any $m_{F,G}$ measurable function and let $Z \subset G^*$ be norming for G. Then

$$\widetilde{m}_{F,G}(f) = \sup \left\{ \int |f| \, d|m_z| : z \in Z_1 \right\}.$$

We next list four properties of the semivariation operator $\widetilde{m}_{F,G}$. These along with others may be found on page 330 in ref 5.

Properties of $\widetilde{m}_{F,G}$ (2.3.31).

(a) $\widetilde{m}_{F,G}$ is subadditive and positively homogeneous on the space of $m_{F,G}$ measurable functions.

(b) $\widetilde{m}_{F,G}(f) \leq \widetilde{m}_{F,G}(g)$ if $|f| \leq |g|$.

(c) $\widetilde{m}_{F,G}(\sum f_n) \leq \sum \widetilde{m}_{F,G}(f_n)$, for every sequence of positive $m_{F,G}$ measurable functions.

(d) If $\widetilde{m}_{F,G}(f_n - f) \longrightarrow 0$, then $f_n \longrightarrow f$ in $m_{F,G}$-measure and there exists a subsequence (f_{n_k}) converging $m_{F,G}$-a.e. to f.

The Space of $m_{F,G}$-Integrable Functions.

We remind the reader that caution (2.3.22) is in effect. Let D be a Banach space.

NOTATION (2.3.32). We denote by $\mathcal{F}_D(m_{F,G})$ the set of all $m_{F,G}$-measurable functions $f : S \longrightarrow D$ such that $\widetilde{m}_{F,G}(f) < \infty$.

REMARK (2.3.33). The vector space $\mathcal{F}_D(m_{F,G})$ is complete with respect to the seminorm $\widetilde{m}_{F,G}$. To establish completeness use property (c) of (2.3.31) and construct the limiting function for a Cauchy sequence in $\mathcal{F}_D(m_{F,G})$ in the same way as is done when one attempts to show that L^1 is complete [17, 23, 46] That $\widetilde{m}_{F,G}$ is a seminorm on $\mathcal{F}_D(m_{F,G})$ is seen from (2.3.31), part (c). The space $\mathcal{F}_D(m_{F,G})$ is called the space of D-valued $m_{F,G}$-integrable functions. Observe also that the definition of $\mathcal{F}_D(m_{F,G})$ implies that for each $z \in Z$, $\mathcal{F}_D(m_{F,G})$ is continuously embedded in $L^1_D(|m_z|)$, the space of D valued m_z integrable functions.

The next proposition is an analogue of proposition (2.3.16).

PROPOSITION (2.3.34). Assume Z is closed in G^*. Let $f : S \longrightarrow D$ be $m_{F,G}$-measurable. Then $\widetilde{m}_{F,G}(f) < \infty$ iff $\int |f| d\, |m_z| < \infty$, for each $z \in Z$.

This proposition leads to the important characterization of $\mathcal{F}_D\ (m_{F,G})$ given in the next corollary.

COROLLARY (2.3.35). If Z is closed in G^*, then

$$\mathcal{F}_D(m_{F,G}) = \bigcap_{z \in Z} L^1_D(|m_z|).$$

NOTATION (2.3.36).

(1) Let $\mathcal{B}_D$, $S_D(\mathcal{R})$ and $S_D(\Sigma)$ denote the subspace of D-valued, $m_{F,G}$-measurable functions which are bounded, $\mathcal{R}$-simple and Σ-simple respectively. It is clear that all these spaces are contained in $\mathcal{F}_D(m_{F,G})$.

(2) If $\mathcal{C}$ is any subspace of $\mathcal{F}_D(m_{F,G})$, denote by $\mathcal{F}_D(\mathcal{C}, m_{F,G})$ the closure of $\mathcal{C}$ in $\mathcal{F}_D(m_{F,G})$, which is, of course, complete.

We now present the formulation of the Vitali and Lebesgue Theorems in terms of $m_{F,G}$ (see Brooks and Dinculeanu [4])

THEOREM (2.3.37). (Vitali). Let (f_n) be a sequence from $\mathcal{F}_D(m_{F,G})$ and let $f : S \longrightarrow D$ be $m_{F,G}$-measurable. If condition (1) below and either of conditions (2a) or (2b) are satisfied, then $f \in \mathcal{F}_D(m_{F,G})$ and $f_n \longrightarrow f$ in $\mathcal{F}_D(m_{F,G})$.

(1) $\widetilde{m}_{F,G}(f_n 1_A) \longrightarrow 0$ as $\widetilde{m}_{F,G}(1_A) \longrightarrow 0$, uniformly in n;

(2a) $f_n \longrightarrow f$ in $m_{F,G}$ measure

(2b) $f_n \longrightarrow f$ pointwise and $m_{F,G}$ is uniformly σ-additive.

Conversely, if $f_n \longrightarrow f$ in $\mathcal{F}_D(\mathcal{B}, m_{F,G})$, then conditions (1) and (2a) are satisfied.

THEOREM (2.3.38). (Lebesgue). Let (f_n) be a sequence from $\mathcal{F}_D(\mathcal{B}, m_{F,G})$, let $f : S \longrightarrow D$ be an $m_{F,G}$-measurable function and $g \in \mathcal{F}_{\mathbb{R}}(\mathcal{B}, m_{F,G})$. If

(1) $|f_n| \leq g$ $m_{F,G}$-a.e. for each n, and any one of the conditions (2a) or (2b) below is satisfied:

(2a) $f_n \longrightarrow f$ in $m_{F,G}$-measure

(2b) $f_n \longrightarrow f$ pointwise and $m_{F,G}$ is uniformly σ-additive

then $f \in \mathcal{F}_D(\mathcal{B}, m_{F,G})$ and $f_n \longrightarrow f$ in $\mathcal{F}_D(m_{F,G})$. The next proposition gives some closure properties of subspaces of $\mathcal{F}_D(m_{F,G})$ (see Brooks and Dinculeanu [5], page 333).

PROPOSITION (2.3.39).

(a) $\mathcal{F}_{\mathbb{R}}(S(\Sigma), m_{F,G}) = \mathcal{F}_{\mathbb{R}}(\mathcal{B}, m_{F,G})$.

(b) If $m_{F,G}$ is uniformly σ-additive, and if $\Sigma = \sigma(\mathcal{R})$, then

$$\mathcal{F}_D(S(\mathcal{R}), m_{F,G}) = \mathcal{F}_D(S(\Sigma), m_{F,G}) = \mathcal{F}_D(\mathcal{B}, m_{F,G}).$$

In particular,

$$\mathcal{F}_D(S(\mathcal{R}), m_{\mathbb{R},E}) = \mathcal{F}_D(S(\Sigma), m_{\mathbb{R},E}) = \mathcal{F}_D(\mathcal{B}, m_{\mathbb{R},E}).$$

Remark (2.3.40). In the theory of integration introduced here, unlike the classical case, the sets $S_D(\mathcal{R})$ and $S_D(\Sigma)$ are not necessarily dense in $\mathcal{B}_D$ for the seminorm $\widetilde{m}_{F,G}$. One look at the Lebesgue theorem (2.3.38) indicates why. The Lebesgue theorem is valid for convergence in measure. However it is not valid, in general, for pointwise convergence, unless $m_{F,G}$ is uniformly σ-additive.

Vector Integration. The Integral $\int f\ dm$

In this section we shall make sense of integrating a Banach valued function with respect to a Banach valued measure (where the measure is an appropriate linear operator). The caution in (2.3.22) is still in force with two modifications; so we shall state it anew here.

Let $m : \Sigma \longrightarrow E \subset L(F,G)$ be a σ-additive measure with finite semi-variation, and assume that m_z is σ-additive for each $z \in Z$, where $Z = G^*$. (2.10)

We will be interested in F-valued functions $f : S \longrightarrow F$ with $f \in \mathcal{F}_F(m_{F,G})$. For ease of notation denote $\mathcal{F}_F(m_{F,G})$ by $\mathcal{F}_{F,G}(m)$.

We now construct the integral. Consider $f \in \mathcal{F}_{F,G}(m)$. Then $f \in L^1_F(|m_z|)$ for each $z \in Z = G^*$. Since $\int f\ dm_z \leq \int |f|\ d|m_z|$ the real number $\int f\ dm_z$ is defined (recall that $m_z : \Sigma \longrightarrow L(F,\mathbb{R}) = F^*$). Define a mapping $H : G^* \longrightarrow \mathbb{R}$ by $H(z) = \int f\ dm_z$. The map H is clearly linear (for $m_{z_1+z_2} = m_{z_1} + m_{z_2}$) and it is bounded. Infact we have by an easy computation that $||H|| \leq \widetilde{m}_{F,G}(f) < \infty$, where $||\cdot||$ denotes the norm of the linear operator H. Thus $H \in G^{**}$. We denote the bounded linear operator H by $\int f\ dm$ and we write $H(z)$ as

$$H(z) \equiv \langle z, H\rangle = \langle z, \int f\ dm\rangle \equiv \int f\ dm_z \equiv \left(\int f\ dm\right)(z). \tag{2.11}$$

Let us now consider the above construction in the case that $E = L(\mathbb{R}, E)$. We then have by proposition (2.3.15) that $\widetilde{m}_{\mathbb{R},E}$ is finite. Hence, as done earlier, we may define the integral $\int \phi\ dm \in E^{**}$, for $\phi \in \mathcal{F}_{\mathbb{R},E}(m)$. For $x^* \in E^*$, we have

$$\langle \int \phi\ dm, x^*\rangle \equiv \int \phi\ dm_{x^*} \equiv \int \phi\ d(x^*m).$$

We note here that m_{x^*} is written as (x^*m).

REMARK (2.3.41).

(1) The integral $\int(\cdot)\,dm$ is continuous on $\mathcal{F}_{F,G}(m)$. We deduce this from the fact that for $f, g \in \mathcal{F}_{F,G}(m)$, we have $\|\int f\ dm - \int g\ dm\| \leq \widetilde{m}_{F,G}(|f-g|)$.

(2) If we consider the situation where $E = L(\mathbb{R}, E)$, then $m_{\mathbb{R},E}$ is uniformly σ-additive. For $m_{\mathbb{R},E} = \{|x^*m|; x^* \in Z\}$ and for any $A \in \Sigma$ we have $|m(A)| = \sup\{|\langle m(A), x^*\rangle|;\ x^* \in E_1^*\}$. In this case the spaces $S_F(\mathcal{R})$ and $S_F(\Sigma)$ are dense in $\mathcal{B}_F$ for the seminorm $\widetilde{m}_{\mathbb{R},E}$ (compare with remark (2.3.40)) as conditions (1) and (2b) of the Lebesgue theorem (2.3.38) hold.

(3) If $m : \Sigma \longrightarrow E \subset L(F,G)$ has finite total variation then $\widetilde{m}_{F,G}$ is finite (for $\widetilde{m}_{F,G} \leq |m|$, see page 52 of Dinculeanu [12]) and hence $m_{F,G}$ is uniformly σ-additive. In this case comments similar to (2) above apply and we have $L_F^1(m) \subset \mathcal{F}_{F,G}(m)$.

(4) When the theory of this section is applied in the next chapter (to the stochastic integral) we will restrict ourselves to functions $f \in \mathcal{F}_{F,G}(m)$ such that (a) $\int f\ dm \in G$ and (b) the stochastic integral $\int_{[0,t]} f\ dm$ is cadlag (part (b) will be explained in chapter 3).

Some Results in Vector Integration.

For what follows the conditions on m in (2.10) are still in force. In this section will state three theorems that will find frequent use in Chapter 3. Prior to doing so we define the indefinite integral. Let $f \in \mathcal{F}_{F,G}(m)$ and denote by $n : \Sigma \longrightarrow G^{**}$ the set function given by $n(A) = \int_A f\ dm \equiv \int 1_A f\ dm$. We call n the indefinite integral of f with respect to m. A common notation for n is fm. Observe that n is a finitely additive measure. The first of the following succession of theorems indicates

conditions under which n is σ-additive. For proofs of these theorems see Brooks and Dinculeanu [5].

THEOREM (2.3.42). Let $f \in \mathcal{F}_{F,G}(m)$. Then fm is σ-additive on Σ in each of the following cases:

(a) $\int_A f\, dm \in G$, for every $A \in \Sigma$; in particular if f is a Σ-step function.

(b) $f \in \mathcal{F}_{F,G}(\mathcal{B}, m)$ and $m_{F,G}$ is uniformly σ-additive; in this case we have $\int f\, dm \in G$ for every $f \in \mathcal{F}_{F,G}(\mathcal{B}, m)$.

(c) G does not contain a copy of c_0; in this case we have $\int f\, dm \in G$.

The next Theorem extends part of Proposition (2.3.15) to functions.

THEOREM (2.3.43). Assume $E \subset L(F,G)$ isometrically and let $m : \Sigma \longrightarrow E \subset L(F,G)$ be a σ-additive measure with finite semivariation $\widetilde{m}_{F,G}$. Then

$$\mathcal{F}_{\mathbb{R}}(m_{F,G}) \subset \mathcal{F}_{\mathbb{R}}(m_{\mathbb{R},E})$$

and $\widetilde{m}_{\mathbb{R},E}(\phi) \leq \widetilde{m}_{F,G}(\phi)$ for $\phi \in \mathcal{F}_{\mathbb{R}}(m_{F,G})$.

THEOREM (2.3.44). Let $m : \Sigma \longrightarrow E \subset L(F,G)$ be a σ-additive measure with finite semivariation $\widetilde{m}_{F,G}$, and let $y \in F$. Then $\mathcal{F}_{\mathbb{R}}(m_{F,G}) \subset \mathcal{F}_{\mathbb{R}}((my)_{\mathbb{R},G})$, $y\mathcal{F}_{\mathbb{R}}(m_{F,G}) \subset \mathcal{F}_F(m_{F,G})$, and for $\phi \in \mathcal{F}_{\mathbb{R}}(m_{F,G})$, we have

$$\begin{aligned} \widetilde{(my)}_{\mathbb{R},G}(\phi) &\leq |y|\widetilde{m}_{F,G}(\phi) \\ \widetilde{m}_{F,G}(\phi y) &= |y|\widetilde{m}_{F,G}(\phi), \end{aligned}$$

and

$$\int \phi y\, dm = \int \phi d(my).$$

If in addition, $\int \phi dm \in E$, then $(\int \phi\, dm)y = \int \phi y\, dm = \int \phi d(my)$.

CHAPTER 3
THE STOCHASTIC INTEGRAL OF PROCESS MEASURES

In this chapter we introduce a theory of stochastic integration by enlarging the class of integrators to p-summable process measures. We recall for the reader that in the Dellacherie-Meyer monograph [10], the most general class of integrators with respect to which they construct the scalar stochastic integral is the class of semimartingales. A very appealing, and simpler, treatment of this theory is given in Protter [41]. The stochastic integration undertaken in this chapter is in a Banach space setting. The history of stochastic integration in Banach spaces goes back to around 1970. During this period there were many authors (such as Pellaumail, Kussmaul, Métivier, Pratelli, Yor and Gravereaux, see [4, 27, 29, 32] for references) who tried to construct an adequate theory of stochastic integration in Banach spaces. There were, however, some shortcomings in these attempts. It was the case that in most of these efforts either the Banach space was too restrictive, as in Kunita [28], or the convergence theorems for a full development of the theory were lacking. These problems were resolved in the theory of stochastic integration developed by Brooks and Dinculeanu [4]. The reason for their success was that they used a sufficiently rich underlying theory of vector integration (see chapter 2, section 3) upon which to base their theory of the stochastic integral. Their construction produced a stochastic integral which could at the same time be viewed as an integral with respect to a measure. The integrators in the Brooks-Dinculeanu theory was the class of semi-summable processes. These processes are more general than semimartingales (see the example after theorem 4.16 [4]).

The theory of stochastic integration that we present here seeks to provide a

broader framework within which the stochastic integral of Brooks and Dinculeanu may be viewed. Defining the stochastic integral in a natural way for elementary processes (see Corollary (3.3.3)) and then proceeding, we are able to embed, for the most part, the theory [4] into our own. Even though our theory enlarges the scope of the stochastic integral it obtains its life from the well established theory [4]. Another source that provided impetus for our theory was the work done by Walsh [47]. Walsh constructed a L^2 theory of stochastic integration for the class of worthy martingale measures (that is, for martingale measures that were dominated by some stochastic measure). The construction provided by Walsh followed, albeit in disguise, the construction of the Ito integral. The construction of our stochastic integral is quite different as it is based on the theory of integration in Banach spaces, which was presented in chapter 2, section 3. If we reduce our process measure to a stochastic process, say, by considering $X_t(B)$ where $B \in \mathcal{B}(\mathbb{R})$ is fixed, we will then obtain the theory in Brooks and Dinculeanu [4].

In our theory we are able to obtain results parallel to those found in the standard theory of the stochastic integral. For instance, we are able to discuss the summability of the stochastic integral, we are able to show that under certain conditions the properties of the integrator carry over to the stochastic integral and we are able to prove the Ito isometry in the case of orthogonal martingale measures.

The primary contribution of our theory is that it allows us to view the stochastic integral as a process measure in the case that the integrator is a process measure. A nice illustration of this statement is the result that if the integrator X is an orthogonal martingale measure then the stochastic integral $H \cdot X$ (see definition (3.2.2)) for H in an appropriate class is also an orthogonal martingale measure (theorem (4.3.6)). After completing the construction of the stochastic integral we shall study some of its properties [4, 8, 10, 16, 19, 20, 24, 27, 33, 48].

3.1 Preliminary Notions for the Stochastic Integral

The process measures $X : \mathbb{R}_+ \times \Omega \times \mathcal{B}(\mathbb{R}_+) \longrightarrow E \subset L(F, G)$ that will be of interest to us in this chapter are those that are p-summable (see chapter 2, section 2). That is, we will only be interested in those process measures X whose corresponding stochastic measures $I_X : \mathcal{R}(\mathcal{R}_P \times \mathcal{B}(\mathbb{R})) \longrightarrow L^p_E$ (where $\mathcal{R}_P$ is the ring of finite (disjoint) unions of predictable rectangles which generate $\mathcal{P}_\infty$) can be extended to a σ-additive L^p_E-valued measure on the σ-algebra $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ with finite semivariation relative to (F, L^p_G) (see definition (2.2.1)). Since I_X is now a σ-additive measure with finite semivariation (relative to (F, L^p_G)), we may return to chapter 2 (section 3) and apply the integration theory presented there to the measure I_X. This is precisely what we shall do. For the comfort of the reader we shall summarize the integration theory in chapter 2 in terms of the symbol I_X. Once this is done we will be able to define the stochastic integral (section 2). The summary of section 3 of chapter 2 in terms of I_X is provided in several steps below.

(1) Extend I_X to $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ by setting $I_X((\{\infty\} \times A) \times B) = 0$, where $A \in \mathcal{F}_\infty$ and $\mathcal{P}_\infty = \mathcal{P} \cup (\{\{\infty\}\} \times \mathcal{F}_\infty)$ with $\mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t$ (see the second paragraph before theorem (2.2.5)).

(2) The symbols Σ, m, E, G in section 2.3 are now replaced by $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$, I_X, L^p_E, L^p_G respectively. The letter Z will now denote a norming space for L^p_G, where Z is such that $Z \subset (L^p_G)^*$. We may, for instance, take Z to be the space of simple functions in $L^q_{G^*}$, where $\frac{1}{p} + \frac{1}{q} = 1$ and G^* is the dual space of G.

(3) For $z \in Z$ denote by $m_z = (I_X)_z : \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}) \longrightarrow F^*$ the σ-additive measure defined as follows. For each $D \in P_\infty \otimes \mathcal{B}(\mathbb{R})$ and $y \in F$ define m_z by

$$\begin{aligned}\langle m_z(D), y\rangle &= \langle m(D)y, z\rangle \\ &\equiv \int_\Omega \langle m(D)(w), z(w)\rangle \; dP(w) \\ &= \int_\Omega \langle I_X(D)(w), z(w)\rangle \; dP(w).\end{aligned}$$

The reader should refer to theorem (2.3.7) and proposition (2.3.13) in the case that this step is not familiar

(4) Let $H : \mathbb{R}_+ \times \Omega \times \mathbb{R} \longrightarrow F$ be a predictable process, that is, H is measurable with respect to the sigma algebra $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$. Then the semivariation of I_X on H with respect to (F, L^p_G) is

$$(\tilde{I}_{F,L^p_G})(H) = \sup\{\int |H| \; d|m_z| : z \in Z_1\}$$

where $m_z = (I_X)_z$ (see proposition (2.3.30)).

(5) For notational convenience we may, on occasion (as was just done in (4) above), omit the letter X from the stochastic measure I_X. That is, it is understood that I and I_X mean the same thing.

(6) Denote by $\mathcal{F}_{F,G}(X) \equiv \mathcal{F}_{F,L^p_G}(X) \equiv \mathcal{F}_F((I_X)_{F,L^p_G})$ the space of F-valued, predictable processes H which satisfy the condition that $(\tilde{I}_{F,L^p_G})(H)$ is finite.

For any subset $\mathcal{C}$ of $\mathcal{F}_F$ $(I_{F,G})$ the notation $\mathcal{F}_F(\mathcal{C}, I_{F,G})$ will denote the closure of $\mathcal{C}$ in $\mathcal{F}_F(I_{F,G})$ [the closure is, of course, under the semivariation norm $\tilde{I}_{F,L^p_G}$.]

(7) The mapping $\theta : \mathcal{F}_{F,G}(X) \longrightarrow Z^*$ given by $\theta(H) = \int H \; dI_X$ is linear and continuous. For any $z \in Z$, $\theta(H)(z)$ is written as $\theta(H)(z) = \langle \int H \; dI, z\rangle = \int H \; dI_z$. The norm of the linear operator θ is bounded by 1. In fact, we have $|\theta(H)|_{Z^*} \leq (\tilde{I})_{F,L^p_g}(H)$, whence $||\theta|| \leq 1$. For details, the reader is directed to

the vector integration section, which appears after remark (2.3.39). We remind the reader that for $H \in \mathcal{F}_{F,G}(X)$, the integral $\int H\, dI_X$ is an element of Z^*.

(8) The previous steps complete the summary of the third section of chapter 2. This step is just for notation. If $H \in \mathcal{F}_{F,G}(X)$ then for each $t \geq 0$ and $B \in \mathcal{B}(\mathbb{R})$ we denote by the symbol $(\int_{[0,t]} H\, dI_X)(B)$ the following:

$$\begin{aligned} \left(\int_{[0,t]} H\, dI_X\right)(B) &= \int_{[0,t]\times B} H\, dI_X \\ &\equiv \int_{[0,t]\times\Omega\times B} H\, dI_X \\ &= \int 1_{[0,t]} 1_B H\, dI_X. \end{aligned}$$

When $t = \infty$ we write

$$\int_{[0,\infty]\times B} H\, dI_X = \int_{[0,\infty)\times B} H\, dI_X = \int_B H\, dI_X. \tag{3.1}$$

The first equality in (3.1) follows because we may write $\int_{[0,\infty]\times B} H\, dI_X$ as the sum $\int_{[0,\infty)\times B} H dI_X + \int_{\{\infty\}\times B} H\, dI_X$, and the second term in this sum is zero by step (1). Now for each $H \in \mathcal{F}_{F,G}(X)$ and $B \in \mathcal{B}(\mathbb{R})$, we obtain a family $(\int_{[0,t]\times B} H\, dI_X)_{t\geq 0}$ of elements in Z^*. The stochastic integral, which we will define in the next section, imposes certain restrictions on this family of elements.

We now present a very useful Lebesgue type theorem which will find frequent use in the future. The proof of this theorem may be found in Brooks and Dinculeanu [4] (theorem 3.1).

THEOREM (3.1.1). Let $(H^n)_{n=1}^{\infty}$ be a sequence of elements from $\mathcal{F}_{F,G}(X)$ such that $|H^n| \leq |H|$ for each n and assume that $H^n \longrightarrow H$ pointwise.

If $(\int H^n dI_X) \in L^p_G$ for each $n \geq 1$ and if the sequence $(\int H^n\, dI_X)_n$ converges pointwise on Ω, weakly in G, then $(\int H\, dI_X) \in L^p_G$, and $\int H^n\, dI_X \longrightarrow \int H\, dI_X$ in the $\sigma(L^p_G, L^q_{G^*})$ topology of L^p_G $(\frac{1}{p} + \frac{1}{q} = 1)$, weakly in G. If $(\int H^n\, dI_X)_n$ converges pointwise, strongly in G, then $\int H^n\, dI_X \longrightarrow \int H\, dI_X$ strongly in L^p_G (and thus in L^1_G).

For the remainder of this chapter we assume that the summable process measure, X, is zero at zero.

3.2 The Stochastic Integral

We recall now the comment we made in the last part of step (8) in section 3.1. We indicated there that in order to view the family $(\int_{[0,t]\times B} H\, dI_X)_{t\geq 0}$ as a stochastic integral certain restrictions must be imposed on it. These restrictions are outlined in definition (3.2.4). The next theorem and definition will pave the way for the definition of the stochastic integral.

THEOREM (3.2.1). Let $B \in \mathcal{B}(\mathbb{R})$ and let $H \in \mathcal{F}_{F,G}(X)$. Assume that $(\int_{[0,t]\times B} H\, dI_X) \in L^p_G(\Omega, \mathcal{F}, P)$. Then $(\int_{[0,t]\times B} H\, dI_X) \in L^p_G(\Omega, \mathcal{F}_t, P)$.

Proof. Assume initially that H is scalar valued, that is, $H \in \mathcal{F}_{\mathbb{R},E}(X)$. If H is of the form $H = 1_{[0_A]}1_C$, where $A \in \mathcal{F}_0$, $C \in \mathcal{B}(\mathbb{R})$, then $\int_{[0,t]\times B} H\, dI_X = \int 1_{\{0\}}1_A 1_{B\cap C}\, dI_X = 1_A X_0(B\cap C) = 0$ (see sections 1 and 2 of chapter 2). Thus $(\int_{[0,t]\times B} H\, dI_X) \in L^p_E(\Omega, \mathcal{F}_t, P)$. Next if H is of the form $H = 1_{(u,v]}1_A 1_C$ then

$$\begin{aligned}\int_{[0,t]\times B} H\, dI_X &= \int 1_{(u\wedge t, v\wedge t]}1_A 1_{B\cap C}\, dI_X \\ &= 1_A[X_{v\wedge t}(B\cap C) - X_{u\wedge t}(B\cap C)] \in L^p_E(\Omega, \mathcal{F}_t, P).\end{aligned}$$

The last equality above is from (2.2).

It now follows that for H of the form $H = 1_D$, $D \in \mathcal{R}$, where $\mathcal{R}$ is the ring of finite (disjoint) unions of cubes of the form $(s,t]\times A\times B$ with $A \in \mathcal{F}_s$ and $B \in \mathcal{B}(\mathbb{R})$, we have

$$\left(\int_{[0,t]\times B} H\, dI_X\right) \in L^p_E(\Omega, \mathcal{F}_t, P).$$

Now let $H = 1_M$, where $M \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$. Denote by $\mathcal{M}$ the set

$$\mathcal{M} = \left\{M \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}) : \left(\int_{[0,t]\times B} 1_M dI_X\right) \in L^p_E(\Omega, \mathcal{F}_t, P)\right\}.$$

We have already seen that $\mathcal{R} \subset \mathcal{M}$. We claim that $\mathcal{M}$ is a monotone class. Let $(M^n)_{n=1}^\infty$ be a sequence in $\mathcal{M}$ such that $M^n \uparrow M$ (the case when $M^n \downarrow M$ is done

the same way). We need to prove that $M \in \mathcal{M}$. Let $z \in (L_E^q)^*$. Since $1_{[0,t]\times B}1_{M^n} \nearrow 1_{[0,t]\times B}1_M$, by monotone convergence we have $1_{[0,t]\times B}1_{M^n} \longrightarrow 1_{[0,t]\times B}1_M$ in $L((I_X)_z)$. Thus,

$\int_{[0,t]\times B} 1_{M^n}\, d(I_X)_z \longrightarrow \int_{[0,t]\times B} 1_M\, d(I_X)_z$, or $\langle \int_{[0,t]\times B} 1_{M^n}\, dI_X, z\rangle \longrightarrow \langle \int_{[0,t]\times B} 1_M\, dI_X, z\rangle$. That is, $\int_{[0,t]\times B} 1_{M^n}\, dI_X \longrightarrow \int_{[0,t]\times B} 1_M\, d(I_X)$ weakly in $L_E^p(\Omega, \mathcal{F}_t, P)$. Now, by hypothesis, $(\int_{[0,t]\times B} 1_{M^n}\, dI_X) \in L_E^p(\Omega, \mathcal{F}_t, P)$ which is a convex subspace of $L_E^p(\Omega, \mathcal{F}, P)$. It follows that $(\int_{[0,t]\times B} 1_M\, dI_X)$ belongs to the weak closure in $L_E^p(\Omega, \mathcal{F}, P)$ of the convex set $L_E^p(\Omega, \mathcal{F}_t, P)$. The space $L^p(\Omega, \mathcal{F}_t, P)$ being convex, its weak closure is the same as its norm closure in $L^p(\Omega, \mathcal{F}, P)$. Hence by completeness of $L_E^p(\Omega, \mathcal{F}_t, P)$ we conclude that $(\int_{[0,t]\times B} 1_M\, dI_X) \in L_E^p(\Omega, \mathcal{F}_t, P)$.

Hence $\mathcal{M}$ is a monotone class and $\mathcal{M} = \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$.

We are now prepared to prove the theorem. Let $H : \mathbb{R}_+ \times \Omega \times \mathbb{R} \longrightarrow F$ be a predictable simple function, say H is of the form $H = \sum_{i=1}^n 1_{M_i} x_i$, with $M_i \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ and $x_i \in F$. Then $\int_{[0,t]\times B} H\, dI_X = \sum_{i=1}^n x_i \int_{[0,t]\times B} 1_{M_i}\, dI_X$ which belongs to $L_G^p(\Omega, \mathcal{F}_t, P)$ (remember that now H is F-valued and I_X is L_E^p valued, where $L_E^p \subset L(F, L_G^p)$). Finally, let $H \in \mathcal{F}_{F,G}(X)$. Since H is predictable (that is, measurable in the Bochner sense) there exists a sequence $(H^n)_{n=1}^\infty$ of predictable simple functions such that $|H^n| \leq |H|$ and $H^n \longrightarrow H$ pointwise. Let $z \in (L_G^p)^*$. By dominated convergence we obtain that $1_{[0,t]\times B}H^n \longrightarrow 1_{[0,t]\times B}H$ in $L_F^1((I_X)_z)$. That is, $\int_{[0,t]\times B} H^n\, d(I_X)_z \longrightarrow \int_{[0,t]\times B} H\, d(I_X)_z$, or

$$\left\langle \int_{[0,t]\times B} H^n\, dI_X, z\right\rangle \longrightarrow \left\langle \int_{[0,t]\times B} H\, dI_X, z\right\rangle.$$

Since $(\int_{[0,t]\times B} H^n\, dI_X) \in L_G^p(\Omega, \mathcal{F}_t, P)$, it follows that $(\int_{[0,t]\times B} H\, dI_X)$ belongs to the weak closure in $L^p(\Omega, \mathcal{F}, P)$ of $L^p(\Omega, \mathcal{F}_t, P)$. Since the weak and norm closures in $L^p(\Omega, \mathcal{F}, P)$ of $L^p(\Omega, \mathcal{F}_t, P)$ are the same we conclude that $(\int_{[0,t]\times B} H\, dI_X) \in L_G^p(\Omega, \mathcal{F}_t, P)$. This completes the proof of the theorem. ∎

DEFINITION (3.2.2). Let $L_{F,G}^1(X)$ denote the Lebesgue space of processes $H \in \mathcal{F}_{F,G}(X)$ which satisfy the two conditions below.

(1) For each $t \geq 0$ the set function $(\int_{[0,t]} H \ dI_X)(\cdot) = \int_{[0,t]\times(\cdot)} H \ dI_X \colon \mathcal{B}(\mathbb{R}) \longrightarrow L^p_G(\Omega, \mathcal{F}, P)$ is a σ-additive measure.

(2) For each $B \in \mathcal{B}(\mathbb{R})$ fixed, the process of representatives $(\int_{[0,t]\times B} H \ dI_X)_{t\geq 0}$ has a cadlag modification.

The elements $H \in L^1_{F,G}(X)$ are said to be integrable with respect to X.

REMARK (3.2.3). Observe that for $B \in \mathcal{B}(\mathbb{R})$ and $H \in L^1_{F,G}(X)$ the element $(\int_{[0,t]\times B} H \ dI_X)$ is a $\mathcal{F}$-measurable and L^p_G valued. In view of theorem (3.2.1) we find that condition (2) in definition (3.2.2) implies that $(\int_{[0,t]\times B} H \ dI_X)$ is $\mathcal{F}_t$-adapted. Thus the requirement for an adapted modification in condition (2) is redundant.

DEFINITION (3.2.4). Let $H \in L^1_{F,G}(X)$. Then we know that for each $B \in \mathcal{B}(\mathbb{R})$ any family of representatives $(\int_{[0,t]\times B} H \ dI_X)_{t\geq 0}$ admits a cadlag, adapted modification. We call this family the Stochastic Integral of H with respect to X and we denote it by $(H \cdot X)$. By the notation $(H \cdot X)_t(B)(w)$ we mean

$$
\begin{aligned}
(H \cdot X)_t(B)(w) &= \left(\int_{[0,t]\times B} H dI_X\right)(w) \\
&\equiv \left(\int_{[0,t]\times\Omega\times B} H \ dI_X\right)(w) \\
&\equiv \left(\int 1_{[0,t]} 1_B H \ dI_X\right)(w)
\end{aligned}
\tag{3.2}
$$

REMARK (3.2.5).

(a) For any element $H \in L^1_{F,G}(X)$ the stochastic integral $(H \cdot X)$ is a process measure.

(b) We caution the reader that as we proceed to prove theorems concerning the stochastic integral we will seldom halt to check that condition (2) of definition (3.2.2) is satisfied. Our negligence will be pardoned by the alert reader who will realize that at the juncture where this property needs proving, it will be transparent. For the benefit of the reader we indicate here the places where

we did not check this condition. They are remark (3.2.6), corollary (3.3.3) and theorem (3.4.6).

REMARK (3.2.6). Let X be a process measure (which need not be p-summable) and let H be a simple F valued process of the form

$$H = 1_{\{0\}\times A_0\times B_0} x_0 + \sum_{1\le i\le n} 1_{(t_i^1,t_i^2]\times A_i\times B_i} x_i \tag{3.3}$$

where $A_0 \in \mathcal{F}_0$, $A_i \in \mathcal{F}_{t_i^1}$ for $i = 1, 2, \dots, n$, $B_i \in \mathcal{B}(\mathbb{R}^+)$ and $x_i \in F$. Then $(H \cdot X)$ is a process measure. We verify this statement. Let $D \in \mathcal{B}(\mathbb{R})$. Using the definition of I_X we have

$$\begin{aligned}(H\cdot X)_t(D)(w) &= \int_{[0,t]\times D} H\, dI_X \\ &= 1_{A_0} X_0(D\cap B_0) + \textstyle\sum_{1\le i\le n} 1_{A_i} x_i [X_{t_i^2\wedge t}(D\cap B_i)(w) \\ &\quad - X_{t_i^1\wedge t}(D\cap B_i)(w)].\end{aligned} \tag{3.4}$$

From (3.4) we see that $(H \cdot X)_t(D)$ is cadlag as X is a process measure. Also condition (1) of definition (3.2.2) is readily seen to be satisfied. Hence $(H \cdot X)$ is a process measure. It follows that simple processes H of the form in (3.3) are in $L^1_{F,G}(X)$.

Our next result is concerned with the left limits of the stochastic integral. The proof for it is similar to that of proposition 3.4 [4]. We provide it here to illustrate the use of theorem (3.1.1).

LEMMA (3.2.7). Let $H \in L^1_{F,G}(X)$, $A \in \mathcal{B}(\mathbb{R})$ and $t \in [0, \infty)$. Then we have:

(1) $(H \cdot X)_{t-}(A) \in L^p_G$ and $(H \cdot X)_{t-}(A) = \int_{[0,t)\times A} H\, dI_X$ w.a.s.

(2) The map $t \longrightarrow (H \cdot X)_t$ is cadlag in L^1_G.

Proof. Let the sequence (t_n) increase to t. Then we have $1_{[0,t_n]} 1_A H \longrightarrow 1_{[0,t)} 1_A H$ pointwise (w). Also, for each n we have:

(a) $|1_{[0,t_n]} 1_A H| \le |1_A H|$ and

(b) $\int_{[0,t_n]\times A} H\, dI_X = (H\cdot X)_{t_n}(A) \in L^p_G$.

Furthermore, since $H \in L^1_{F,G}(X)$, we have

$$(H\cdot X)_{t_n}(A) \longrightarrow (H\cdot X)_{t_-}(A) \quad \text{(the convergence is in } G). \tag{3.5}$$

Upon applying theorem (3.1.1) we deduce that $\int_{[0,t_n]\times A} H dI_X \longrightarrow \int_{[0,t)\times A} H\, dI_X$ in L^1_G. Hence there exists a subsequence (t_{n_k}) such that $(H\cdot X)_{t_{n_k}}(A) \longrightarrow \int_{[0,t)\times A} H\, dI_X$ pointwise. However, from (3.5), we know that $(H\cdot X)_{t_n}(A) \longrightarrow (H\cdot X)_{t_-}(A)$ pointwise. It follows that $(H\cdot X)_{t_-}(A) = \int_{[0,t)\times A} H\, dI_X$. Observe also that by completeness of the space L^p_G we have $(H\cdot X)_{t_-}(A) \in L^p_G$. This proves statement (1) of the theorem.

As for statement (2) we have already seen that $(H\cdot X)_{t_n}(A) \longrightarrow (H\cdot X)_{t_-}(A)$ in $L^p_G \subset L^1_G$. That is the left limit of $(H\cdot X)_t(A)$ exists in L^p_G. To establish right continuity let $s_n \downarrow s$. Then $1_{[0,s_n]}1_A H \longrightarrow 1_{[0,s]}1_A H$ pointwise and $|1_{[0,s_n]}1_A H| \leq |1_A H|$. Applying theorem (3.1.1) we conclude that $\int_{[0,s_n]\times A} H\, dI_X \longrightarrow \int_{[0,s]\times A} H\, dI_X$ in L^1_G. That is, $(H\cdot X)_{s_n}(A) \longrightarrow (H\cdot X)_s(A)$ in L^1_G. This proves statement (2). ■

Remark (3.2.8). Statement (1) in theorem (3.2.7) holds even when $t = \infty$. The proof is the same once we recall two facts: (a) we may write $\int_{[0,\infty]\times A} H\, dI_X$ as $\int_{[0,\infty)\times A} H\, dI_X + \int_{\{\infty\}\times A} H\, dI_X$ and (b) $I_X(\{\infty\}\times\Omega\times A) = 0$.

3.3 Elementary Processes and their Stochastic Integrals

We have already seen that for a simple predictable F-valued elementary process H of the form in (3.3), that is

$$H = 1_{\{0\}\times A_0\times B_0} x_0 + \sum_{1\leq i\leq n} 1_{(t_i^1, t_i^2]\times A_i\times B_i} x_i,$$

$H\cdot X$ is a process measure, which is the same as saying that $H \in L^1_{F,G}(X)$ (see theorem (3.2.1), definition (3.2.2) and remark (3.2.3).

The next theorem and corollary will show that the following more general class

of elementary processes of the form

$$H = H_0 1_{\{0\}} 1_{A_0} + \sum_{1 \le i \le n} H_i 1_{((T_i, T_{i+1}]]} 1_{A_i}, \tag{3.6}$$

are in $L^1_{F,G}(X)$. In (3.6) the family $(T_i)_{0 \le i \le n+1}$ is an increasing family of stopping times with $T_0 = 0$, the H_i, $i = 0, \dots, n$ are bounded random variables which are $\mathcal{F}_{T_i}$-measurable and $A_i \in \mathcal{B}(\mathbb{R})$ for $i = 0, \dots, n$. The proofs of both theorem (3.3.2) and corollary (3.3.3) below are carried out along the same lines as the proof of proposition (3.5) in Brooks and Dinculeanu [4] and hence will be omitted. We remark here that theorem (2.2.5) is required in the proof of theorem (3.3.2). We will need the following proposition prior to stating this theorem.

PROPOSITION (3.3.1). Let X be a (summable) process measure and let T be a stopping time. Then, we have

(1) X^T is a process measure, and

(2) $X_T(\cdot)$ is a L^p_E-valued σ-additive measure.

Proof. We prove (1) first. Let us verify condition (a) of definition (2.1.1). Let $A \in \mathcal{B}(\mathbb{R})$ and $t \in [0, \infty]$. Then $X^T_t(A) = X_{T \wedge t}(A)$ which is $\mathcal{F}_{T \wedge t} \subset \mathcal{F}_t$-measurable (see, for example, page 85 in [27]). It remains for us to verify condition (b) of definition (2.1.1). But this is immediate from theorem (2.2.5) (b) and the fact that X is summable. For we have $X_{T \wedge t}(A) = I_X([[0, T \wedge t]] \times A)$ and I_X is a L^p_E-valued σ-additive measure on $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ (see definition (2.2.1)). Thus for $[[0, T \wedge t]]$ fixed, $I_X([[0, T \wedge t]] \times (\cdot)) : \mathcal{B}(\mathbb{R}) \longrightarrow L^p_E$ is a measure; that is $X_{T \wedge t}(\cdot)$ is a L^p_E-valued σ-additive measure. This proves statement (1). Statement (2) is proved the same way except that now we replace $T \wedge t$ by T. ∎

We are now ready to state the promised theorem and corollary.

THEOREM (3.3.2). Let S and T be stopping times with $S \le T$ and suppose that $h : \Omega \longrightarrow F$ is a bounded $\mathcal{F}_S$-measurable random variable. Assume that $A \in \mathcal{B}(\mathbb{R})$. The following two statements hold.

(1) $\int h1_{((S,T]]}1_A \, dI_X = h(X_T(A) - X_S(A))$.

(2) If S is predictable and h is $\mathcal{F}_S$-measurable, then

(a) $\int h1_{[[S,T))}1_A \, dI_X = h(X_T(A) - X_{S-}(A))$ and,

(b) $\int h1_{[[S]]}1_A \, dI_X = h\Delta X_S(A)$.

COROLLARY (3.3.3). Assume that H is an elementary process of the form in (3.6). Then $H \in L^1_{F,G}(X)$, that is, H is integrable with respect to X. The stochastic integral $(H \cdot X)$ can be computed pathwise as follows:

$$
\begin{aligned}
(H \cdot X)_t(A) &= H_0 X_0(A \cap A_0) + \sum_{1 \le i \le n} H_i(X_{T_{i+1} \wedge t}(A \cap A_i) \\
&\quad - X_{T_i \wedge t}(A \cap A_i)) \\
&\equiv H_0 X_0(A \cap A_0) + \sum_{1 \le i \le n} H_i(X_t^{T_{i+1}}(A \cap A_i) \\
&\quad - X_t^{T_i}(A \cap A_i))
\end{aligned}
\tag{3.7}
$$

REMARK (3.3.4).

(a) Observe that by virtue of proposition (3.3.1) the process $H \cdot X$ in (3.7) is indeed a process measure.

(b) It will be shown in section 4 that if X is summable then so is X^T (see theorem (3.4.5)).

3.4 Stopped Process Measures and Summability of the Stochastic Integral

We will continue to assume (for the remainder of chapter 3) that X is a p-summable process measure. In this section we will study some properties of stopped process measures. We have already shown that if X is a summable process measure then for any stopping time T, X^T is a process measure. Infact X^T is also a summable process measure- -this fact will be established in theorem (3.4.5). Our main goal in this section is to show that if $H \in L^1_{F,G}(X)$, that is, if $H \cdot X$ is a process measure,

then $H \cdot X$ is also summable. We will then be able to show that the (summable) process measure $(H \cdot X)^T$ is the same object as the summable process measure $H \cdot X^T$. This, of course, is nothing new in the standard theory of stochastic integration. The novelty comes from the fact that the equality $(H \cdot X)^T = H \cdot X^T$ is now as process measures. As preparation for the proof of these results we first generalize theorem (3.3.2). Remember that throughout this chapter the process measure X is assumed to be p-summable

PROPOSITION (3.4.1). Let S and T be stopping times with $S \leq T$. Suppose that either condition (I) or condition (II) below is in force.

Condition (I). $h : \Omega \longrightarrow \mathbb{R}$ is bounded, $\mathcal{F}_S$-measurable and $H \in \mathcal{F}_{F,G}(X)$.

Condition (II). $h : \Omega \longrightarrow F$ is bounded, $\mathcal{F}_S$-measurable and $H \in \mathcal{F}_{\mathbb{R}}((I_X)_{F,G})$.

Statement (a) below holds under condition (I) and statement (b) below holds under condition (II).

(a) If $\int 1_{((S,T]]} H \, dI_X \in L^p_G$ then

$$\int h 1_{((S,T]]} H \, dI_X = h \int 1_{((S,T]]} H \, dI_X \tag{3.8(a)}$$

(b) If $\int 1_{((S,T]]} H \, dI_X \in L^p_E$ then

$$\int h 1_{((S,T]]} H \, dI_X = h \int 1_{((S,T]]} H \, dI_X \tag{3.8(b)}$$

Proof. We prove (a) first.

Assume now that condition (I) holds. Consider $H \in \mathcal{F}_{F,G}(X)$ of the form $H = 1_{((s,t]] \times A} 1_B y$, where $A \in \mathcal{F}_s$ and $B \in \mathcal{B}(\mathbb{R})$ and $y \in F$. By an application of theorem (3.3.2) we have

$$\begin{aligned} \int h 1_{((S,T]]} H \, dI_X &= \int h 1_{((S \vee s, T \wedge t]]} 1_A 1_B y \, dI_X \\ &= \int (h 1_A) 1_{((S \vee s, T \wedge t]]} 1_B y \, dI_X \\ &= (h 1_A) y [X_{T \wedge t}(B) - X_{S \vee s}(B)]. \end{aligned} \tag{3.9}$$

The last term in (3.9) can be written as

$$\begin{aligned} hy[X_{(T\wedge t)_A}(B) - X_{(S\vee s)_A}(B)] &= hI_X(y[[T\wedge t)_A, (S\vee s)_A]] \times B) \\ &= h\int y1_{(((T\wedge t)_A,(S\vee s)_A]]}1_B\, dI_X \\ &= h\int 1_{((S,T]]}1_{(s,t]\times A}1_B y\, dI_X \\ &= h\int 1_{((S,T]]}H\, dI_X \end{aligned} \tag{3.10}$$

Since h is real valued and bounded the last term in (3.10) is in L^p_G (see theorem (2.2.5)).

It follows that for any set D in $\mathcal{R}$ (the ring generating the σ-algebra $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$) we still have

$$\int h1_{((S,T]]}1_D y\, dI_X = h\int 1_{((S,T]]}1_D y\, dI_X \in L^p_G. \tag{3.11}$$

Now let $z \in L^q_{G_*}$. We then have

$$\int h1_{((S,T]]}1_D y\, d(I_X)_z = \int 1_{((S,T]]}1_D y\, d(I_X)_{hz}. \tag{3.12}$$

The equality in (3.12) is obtained as follows:

$$\begin{aligned} \int h1_{((S,T]]}1_D y\, d(I_X)_z &= \left\langle \int h1_{((S,T]]}1_D y\, dI_X, z\right\rangle \\ &= \left\langle h\int 1_{((S,T]]}1_D y\, dI_X, z\right\rangle \\ &= \left\langle \int 1_{((S,T]]}1_D y\, dI_X, hz\right\rangle \\ &= \int 1_{((S,T]]}1_D y\, d(I_X)_{hz}. \end{aligned} \tag{3.13}$$

The third equality in (3.13) is valid because h is real valued and bounded, thus $hz \in L^q_{G^*}$.

The equality in (3.12) remains valid for sets D in $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$. We prove this by a monotone class argument. Let $\mathcal{M} = \{D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$: (3.12) holds for $D\}$. We have already seen that $\mathcal{R} \subset \mathcal{M}$. Let $(D_n)_{n=1}^\infty$ be a sequence in $\mathcal{M}$ with $D_n \uparrow D$ and we show that $D \in \mathcal{M}$. Let $z \in L^q_{G^*}$ and assume condition (I). Then we

have $h1_{((S,T]]}1_{D_n}y \longrightarrow h1_{((S,T]]}1_D y$ pointwise and $|h|1_{((S,T]]}1_{D_n}|y| \leq c|y|1_{((S,T]]}1_{D_n} \in L^1(I_X)_z)$. The constant c bounds h, that is $|h| \leq c$. Recall that since X is summable I_X has finite semivariation (relative to (F,G)) and hence $|(I_X)_z|((S,T]] \times D_n) < \infty$. By dominated convergence we have

$$\int h1_{((S,T]]}1_{D_n}y \; d(I_X)_z \longrightarrow \int h1_{((S,T]]}1_D y \; d(I_X)_z. \tag{3.14}$$

A similar application of the dominated convergence theorem gives us

$$\int 1_{((S,T]]}1_{D_n}y d(I_X)_{hz} \longrightarrow \int 1_{((S,T]]}1_D y \; d(I_X)_{hz}. \tag{3.15}$$

We remark here that the verification of (3.15) uses the fact that

$$|(I_X)_{hz}|(D) \leq c|(I_X)_{hz}|(D) \tag{3.16}$$

where c is a constant that bounds h. Combining (3.14) and (3.15) we obtain (3.12) for sets D in $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$. A similar argument applies when the sequence D_n decreases to D. Hence $\mathcal{M}$ is a monotone class.

It follows that for any predictable, simple process H, we have

$$\int h1_{((S,T]]}H d(I_X)_z = \int 1_{((S,T]]}H \; d(I_X)_{hz}. \tag{3.17}$$

Equality (3.17) remains valid for any $H \in \mathcal{F}_{F,G}(X)$. An application of the dominated convergence theorem (as was done for (3.14) and (3.15)) easily verifies this. Assume now that $(\int 1_{((S,T]]}H \; dI_X) \in L^p_G$. Then we have

$$\begin{aligned} \left\langle h\int 1_{((S,T]]}H \; dI_X, z\right\rangle &= \left\langle \int 1_{((S,T]]}H \; dI_X, hz\right\rangle \\ &= \int 1_{((S,T]]}H \; d(I_X)_{hz} \\ &= \int h1_{((S,T]]}H \; dI_{X_z} \\ &= \left\langle \int h1_{((S,T]]}H \; dI_X, z\right\rangle \end{aligned} \tag{3.18}$$

The third equality in (3.18) is by (3.17). Since $L^q_{G^*}$ is norming for both L^p_G and $(L^q_{G^*})^*$, we obtain the equality in (3.8(a)). This proves (a). We next prove (b).

Assume condition (II). Let $H : \mathbb{R}_+ \times \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be predictable with $\widetilde{I}_{F,G}(H) < \infty$, that is,

$$H \in \mathcal{F}_{\mathbb{R}}(I_{F,G}) \subset \mathcal{F}_{\mathbb{R}}(I_{\mathbb{R},E}) \equiv \mathcal{F}_{\mathbb{R},E}(X). \tag{3.19}$$

The inclusion in (3.19) is valid because $\widetilde{I}_{\mathbb{R},E} \leq \widetilde{I}_{F,G}$, see proposition (2.3.15). Assume also that $(\int 1_{((S,T]]} H \, dI_X) \in L^p_E$ (see (b) of the present proposition). Let h in condition (II) be of the form $h = ky$, where $k : \Omega \longrightarrow \mathbb{R}$ is $\mathcal{F}_S$-measurable and $y \in F$. Then $(\int 1_{((S,T]]} H \, dI_X) y \in L^p_G$. By theorem (2.3.42) we have

$$y \left(\int 1_{((S,T]]} H \, dI_X \right) = \int 1_{((S,T]]} H y \, dI_X. \tag{3.20}$$

The following equalities are then obtained.

$$\int h 1_{((S,T]]} H \, dI_X = k \int 1_{((S,T]]} H y \, dI_X = h \int 1_{((S,T]]} H \, dI_X \tag{3.21}$$

The first and second equalities in (3.21) are realized via (3.8(a)) and (3.20) respectively. It follows that (3.21) remains valid for any $\mathcal{F}_S$-measurable simple function h. For the general case let $h : \Omega \longrightarrow F$ be $\mathcal{F}_S$-measurable and bounded. Then there is a sequence $(h_n)_{n=1}^{\infty}$ of $\mathcal{F}_S$-measurable simple functions such that $h_n \longrightarrow h$ with $|h_n| \leq |h|$. Observe that $h_n 1_{((S,T]]} H \in \mathcal{F}_{\mathbb{R}}(I_{F,G})$ and that $|h_n|_{((S,T]]} H| \leq |h| 1_{((S,T]]} |H|$ (note: h is bounded). We have

$$\int h_n 1_{((S,T]]} H \, dI_X = h_n \left(\int 1_{((S,T]]} H \, dI_X \right) \longrightarrow h \left(\int 1_{((S,T]]} H \, dI_X \right). \tag{3.22}$$

The equality in (3.22) results from (3.21) and the convergence is pointwise (w) in G. Then by the Lebesgue theorem (theorem (3.1.1)) we have

$$\int h_n 1_{((S,T]]} H \, dI_X \longrightarrow \int h 1_{((S,T]]} H \, dI_X \tag{3.23}$$

in L^1_G. Combining (3.22) and (3.23) we obtain (3.8(b)). This proves statement (b) as well as the theorem. ■

REMARK (3.4.2).

(a) Proposition (3.4.1) is just a result on integration; observe that H was not assumed to be in $L^1_{F,G}(X)$ or in $L^1_{\mathbb{R}}((I_X)_{F,G})$. The usefulness of this proposition is in its application to the next two results.

(b) In the case that S is a predictable stopping time, h is $\mathcal{F}_{S_-}$-measurable and H is integrable with respect to X, we have a result similar to that in proposition (3.4.1). Now we have

$$\int h1_{[[S,T]]}H\,dI_X = h\int 1_{[[S,T]]}H\,dI_X. \tag{3.24}$$

The result in (3.24) is easily obtained from proposition (3.4.1) by approximating S by predictable times S_n from below, that is $S_n \uparrow S$.

The next result says that if $H \in L^1_{F,G}(X)$ then under certain conditions the process measure $H \cdot X$ (see remark (3.2.3)) is summable.

THEOREM (3.4.3). Let $H \in L^1_{F,G}(X)$ and assume that either condition (a) or condition (b) is in force.

Condition (a). G does not contain an isomorphic copy of c_0.

Condition (b). For every set $D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ we have $\int_D H\,dI_X \in L^p_G$.

Then we have:

(1) For every set $D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$, $I_{H\cdot X}(D) = \int_D H\,dI_X$.

(2) $(\tilde{I}_{H\cdot X})_{\mathbb{R},L^p_G}(D) \le (\tilde{I}_X)_{F,L^p_G}(1_D H)$, for $D \in \mathcal{R}$.

(3) $H \cdot X$ is p-summable relative to $(\mathbb{R}, L^p_G)$.

Proof. Assume condition (a). Before verifying statements (1)-(3) above we show that the hypothesis and condition (a) together imply condition (b). Let $(s,t] \times A \times B$ be a predictable cube with $A \in \mathcal{F}_s$ and $B \in \mathcal{B}(\mathbb{R})$. Since $H \in L^1_{F,G}(X)$ we have that $(H\cdot X)_t(B) - (H\cdot X)_s(B)$ is L^p_G valued, and it is in fact a measure in B. The following expression is thus also L^p_G-valued:

$$1_A[(H\cdot X)_t(B) - (H\cdot X)_s(B)]. \tag{3.25}$$

We may write (3.25) as $1_A[\int_{(s,t]\times B} H\ dI_X]$. By proposition (3.4.1) (use condition (I) there) the last term is the same as

$$\int 1_{(s,t]} 1_A 1_B H\ dI_X \equiv \int_{(s,t]\times A\times B} H\ dI_X. \tag{3.26}$$

Hence $\int_{(s,t]\times A\times B} H\ dI_X \in L^p_G$. Now let $\mathcal{M}$ denote the set

$$\mathcal{M} = \left\{ D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}) : \int_D H\ dI_X \in L^p_G \right\}.$$

We have just seen that $\mathcal{R} \subset \mathcal{M}$, where $\mathcal{R}$ is the disjoint (finite) union of predictable cubes which generates $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$. Let (D_n) be a sequence in $\mathcal{M}$ such that $D_n \downarrow D$. We assert that $D \in \mathcal{M}$. Since $c_0 \not\subset G$, it follows also that $c_0 \not\subset L^p_G$ [1, 4, 5, 30] and thus the set of measures $(I_X)_{F,G} = \{|(I_X)_z| : z \in (L^q_{G^*})\}$ is uniformly σ-additive (proposition (2.3.20)). Then by the dominated convergence theorem for I_X (theorem (2.3.36)) we conclude that $1_{D_n} H \longrightarrow 1_D H$ in $\mathcal{F}_F((I_X)_{F,G})$.

It follows that

$$\begin{aligned} &|| \int 1_{D_n} H\ dI_X - \int 1_D H\ dI_X ||_{L^p_G} \equiv || \int 1_{(D_n \Delta D)} H\ dI_X ||_{L^p_G} \\ &\longrightarrow 0 \text{ as } n \longrightarrow \infty \text{ as} \\ &\left\| \int 1_{(D_n \Delta D)} H\ dI_X \right\|_{L^p_G} \le (\tilde{I}_X)_{F,L^p_G}(1_{(D_n\Delta D)} H) \longrightarrow 0 \end{aligned} \tag{3.27}$$

(the reader may want to consult the details under the heading "vector integration: The integral $\int f\ dm$", which appears after remark (2.3.38) in chapter 2, in the case that (3.27) causes some confusion).

From (3.27) we conclude that $\int_D H\ dI_X \in L^p_G$. The same argument works if the sequence (D_n) in $\mathcal{M}$ increases to D. Hence $\mathcal{M}$ is a monotone class and we have $\mathcal{M} = \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$. That is, $\int_D H\ dI_X \in L^p_G$, for $D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ and this is condition (b). It follows now, form Theorem (2.3.42) that $d(HI_X)$ is a σ-additive measure on $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$.

We now prove statements (1) and (2) for sets in $\mathcal{R}$. As earlier, consider predictable cubes of the form: (a) $(s,t] \times A \times B$, with $A \in \mathcal{F}_s$ and $B \in \mathcal{B}(\mathbb{R})$ and (b) $\{0\} \times C \times D$

with $C \in \mathcal{F}_0$ and $D \in \mathcal{B}(\mathbb{R})$. These are the generators of $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$. Let us now verify that statement (1) is true for sets of the form in (a) and (b). For the set in (a) we have

$$\begin{aligned}
\int_{(s,t]\times A\times B} H\ dI_X &= \int 1_{(s,t]}1_A 1_B H\ dI_X \\
&= 1_A \int 1_{(s,t]}1_B H\ dI_X = 1_A \int_{(s,t]\times B} H\ dI_X \\
&= 1_A((H\cdot X)_t(B) - (H\cdot X)_s(B)) \\
&= I_{H\cdot X}((s,t]\times A\times B)
\end{aligned} \tag{3.28}$$

The second equality in (3.2.8) is valid by proposition (3.4.1) (condition (I)). The last equality in (3.28) is by definition (see (2.3)). For the set in (b) we have

$$\begin{aligned}
\int_{\{0\}\times C\times D} H\ dI_X &= \int 1_{\{0\}}1_C 1_D H\ Id_X \\
&= 1_C \int_{\{0\}\times D} H\ dI_X \\
&= 1_C(H\cdot X)_0(D) \\
&= I_{H\cdot X}(\{0\}\times C\times D).
\end{aligned} \tag{3.29}$$

The reasoning for (3.29) is the same as that for (3.28). It now follows that

$$\int_D H\ dI_X = I_{H\cdot X}(D) \text{ for } D \in \mathcal{R} \tag{3.30}$$

Since $\int_{(\cdot)} H\ dI_X$ is a L^p_G valued measure on $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ (use theorem (2.3.42) (a)) and as $I_{H\cdot X}$ is a σ-additive measure on $\mathcal{R}$ (from (3.30)) we deduce by a monotone class argument that $I_{H\cdot X}(\cdot)$ can be extended to a L^p_G-valued σ-additive measure on $\mathcal{P}_\infty \otimes B(\mathbb{R})$ and that

$$I_{H\cdot X}(D) = \int_D H\ dI_X \text{ for every } D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}).$$

This proves (1). Let us verify (2) for sets of the form in (a) and (b). For a set in (a)

we have

$$
\begin{aligned}
&(\tilde{I}_{H\cdot X})_{\mathbb{R},L^p_G}((s,t]\times A\times B)=\\
&=\sup_{\substack{(f_i)\subset\mathbb{R},|f_i|\le 1\\ \cup_J((s_i,t_i]\times A_i\times B_i)\\ \subset(s,t]\times A\times B,\\ J\text{ is a finite index set}}} \left\|\sum_{i\in J} I_{H\cdot X}((s_i,t_i]\times A_i\times B_i)f_i\right\|_{L^p_G}\\
&=\sup_{\substack{(f_i)\subset\mathbb{R},|f_i|\le 1\\ \cup_J((s_i,t_i]\times A_i\times B_i)\\ \subset(s,t]\times A\times B,\\ J\text{ is a finite index set}}} \left\|\sum_{i\in J}\left(\int_{(s_i,t_i]\times A_i\times B_i} H\,dI_X\right)f_i\right\|_{L^p_G} \qquad (3.31)\\
&=\sup_{\substack{(f_i);|f_i|\le 1\\ \cup_J(s_i,t_i]\times A_i\times B_i\subset(s,t]\times A\times B}}\ \sup_{z\in(L^q_{G^*})}\left|\left\langle\sum_{i\in J}\int_{(s_i,t_i]\times A_i\times B_i} Hd\ I_X f_i, z\right\rangle\right|\\
&\le(\tilde{I}_X)_{F,L^p_G}(1_{(s,t]\times A\times B}H)\le\tilde{I}_{F,L^p_G}(H)<\infty
\end{aligned}
$$

The second equality in (3.31) is justified by statement (1) and the last term in (3.31) is finite because $H\in L_{F,G}(X)$. The computation in (3.31) is still true for sets of the form (b). It follows that statement (2) holds for sets D in $\mathcal{R}$. Now observe that the σ-additive measure $I_{H\cdot X}$ has bounded semivariation relative to $(\mathbb{R}, L^p_G)$ on $\mathcal{R}$. Hence by proposition (2.3.17) $I_{H\cdot X}$ has bounded semivariation relative to $(\mathbb{R}, L^p_G)$ on $\mathcal{P}_\infty\otimes\mathcal{B}(\mathbb{R})$.

Finally, statement (3) follows immediately from statements (1) and (2). This completes the proof of the theorem. ■

REMARK (3.4.4).

(i) From the proof of theorem (3.4.3) we see that if $H\in L^1_{F,G}(X)$ then for any predictable rectangle of the form $(s,t]\times A\times B$, where $A\in\mathcal{F}_s$ and $B\in\mathcal{B}(\mathbb{R})$, we have

$$\int_{(s,t]\times A\times B} H\ dI_X = 1_A[(H\cdot X)_t(B)-(H\cdot X)_s(B)]\in L^p_G.$$

However the mere fact that $H\in L^1_{F,G}(X)$ does not imply that $\int_D H\ dI_X\in L^p_G$ for an arbitrary set D in $\mathcal{P}_\infty\otimes\mathcal{B}(\mathbb{R})$. This was the reason why we needed to impose either condition (a) or condition (b) in the theorem.

(ii) If $H\in L^1_{\mathbb{R},L^p_E}(X)$ and either one of the conditions (a) or (b) below hold then we can show that $H\cdot X$ is p-summable relative to (F, L^p_G). The conditions are:

(a) $c_0 \not\subset E, G$ and (b) $(\int_D H\ dI_X) \in L^p_E$ for every D in $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$. The proof is the same except that we now have:

$$(\tilde{I}_{H\cdot X})_{F,L^p_G}(D) = ((\tilde{I})_X)_{F,L^p_G}(1_D H) \quad \text{for} \quad D \in \mathcal{R}.$$

(iii) Note that if the space G is reflexive then $(L^p_G)^{**} = L^p_G$, hence the linear operator $\int H dI_X$ belongs to L^p_G. Then condition (b) in the theorem is immediately satisfied (apply theorem (2.3.42)) and there is no need for condition (a).

THEOREM (3.4.5). Let X be a summable process measure and let T be a stopping time.

(a) X^T is p-summable relative to (F, L^p_G).
For any $A \in \mathcal{P}_\infty$ and $B \in \mathcal{B}(\mathbb{R})$ we have

(i) $X^T(B) = (1_{[[0,T]]} \cdot X)(B)$.

(ii) $I_{X^T}(A \times B) = I_X(([[0,T]] \cap A) \times B)$.

(b) Let $H \in \mathcal{F}_{F,L^p_G}(X)$. Then

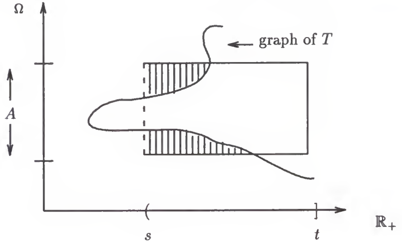
$$(\tilde{I}_{X^T})_{F,L^p_G}(H) = (\tilde{I}_X)_{F,L^p_G}(1_{[[0,T]]}H).$$

Proof. (a) We have already seen in proposition (3.3.1) that X^T is a process measure. For $t \in [0,\infty]$, we know from theorem 2.2.5 (b) that $X^T_t(B) = X_{T\wedge t}(B) = I_X([[0,T \wedge t]] \times B)$. Now consider the predictable function $H = 1_{[[0,T]]}1_B$. Then $H \in L^1_{F,G}(X)$. Infact from corollary (3.3.3) we know that $1_{((0,T]]}1_B \in L^1_{F,G}(X)$. Observe that we may write H as $H = 1_{\{0\}\times\Omega}1_B + 1_{((0,T]]}1_B$ and as $1_{\{0\}\times\Omega}1_B$ is clearly in $L^1_{F,G}B(X)$ it follows that $H \in L^1_{F,G}(X)$. Computing $\int_{[0,t]\times B} H\ dI_X$ we have $\int_{[0,t]\times B} H\ dI_X = \int 1_{[[0,T\wedge t]]}1_B\ dI_X = I_X([[0,T \wedge t]] \times B)$. That is, $(H \cdot X)_t(B) = (1_{[[0,T]]} \cdot X)(B) = I_X([[0,T \wedge t]] \times B)$. Comparing the computations for $(H \cdot X)_t(B)$ and $X^T_t(B)$ we obtain (i).

As for (ii) consider a rectangle $(s,t] \times A$, $A \in \mathcal{F}_s$ in $\mathcal{P}$. We have

$$\begin{aligned} I_{X^T}((s,t] \times A \times B) &= 1_A[X_t^T(B) - X_s^T(B)] \\ &= 1_A[X_{T\wedge t}(B) - X_{T\wedge s}(B)] \\ &= I_X[\{((s,t] \times A) \cap [[0,T]]\} \times B] \end{aligned} \tag{3.32}$$

The third equality in (3.32) may easily be verified by direct (setwise) computation. The picture below will perhaps make this equality transparent.



(The shaded region represents the set $((s,t] \times A) \cap [[0,T]]$.)

This gives (ii) for sets in $\mathcal{P}_\infty$ of the form $(s,t] \times A$, where $A \in \mathcal{F}_s$. Thus we have

$$I_{X^T}(C \times B) = I_X((C \cap [[0,T]]) \times B) \tag{3.33}$$

for sets C in $\mathcal{R}_\mathcal{P}$, where $\mathcal{R}_\mathcal{P}$ is the ring of disjoint (finite) unions of rectangles which generate $\mathcal{P}_\infty$. Now since I_X is a σ-additive measure we have, in particular, that for each fixed B the right hand side of (3.33) is a σ-additive measure on $\mathcal{R}_\mathcal{P}$. Thus for B fixed $I_{X^T}(\cdot \times B) : \mathcal{R}_\mathcal{P} \longrightarrow L_E^p$ is a σ-additive measure and as I_X is a σ-additive measure on $\mathcal{P}_\infty$ it follows that (3.33) can be extended to sets in $\mathcal{P}_\infty$ (by a monotone class argument). That is, we have:

$$I_{X^T}(A \times B) = I_X((A \cap [[0,T]]) \times B) \text{ for } A \in \mathcal{P}_\infty. \tag{3.34}$$

This gives (ii). Finally to show that X^T is summable observe that (3.34) is true for sets in $\mathcal{R}$ (where $\mathcal{R}$ now is the ring generating $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$). Then use the fact

that X is summable to deduce that I_{X^T} can be extended to a σ-additive measure on $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$. Also I_{X^T} has bounded semivariation on $\mathcal{R}$ (because I_X does). Thus we may conclude, by proposition (2.3.17) that X^T is p-summable. This proves (a). We now prove (b). If we compute the semivariation of H with respect to X^T (relative to (F, L^p_G)) in terms of proposition (2.3.30) we see that to establish the equality in (b) it suffices to establish (3.35) below.

$$\int |H| \, d|(I_{X^T})_z| = \int 1_{[[0,T]]}|H| \, d|(I_X)_z| \text{ for each } z \in (L^q_{G^*})_1. \tag{3.35}$$

We verify (3.35) in steps. We recall first the measure $(I_X)_z$. Note that $I_X : \mathcal{P} \longrightarrow L^p_E \subset L(F, L^p_G)$ is a L^p_E-valued measure and $(I_X)_z : \mathcal{P} \longrightarrow L(F, \mathbb{R}) = F^*$ (refer to section (2.3) concerning the measure $(I_X)_z = m_z$, where $I_X = m$). A similar comment applies to I_{X^T}.

Step (1). Let $(s,t] \times A \times B$, with $A \in \mathcal{F}_s$ and $B \in \mathcal{B}(\mathbb{R})$ and let $f \in F$. We have

$$\begin{aligned} (I_{X^T})_z((s,t] \times A \times B)(f) &= \langle I_{X^T}((s,t] \times A \times B)(f), z\rangle \\ &= \langle I_X((((s,t] \times A) \cap [[0,T]])) \times B)(f), z\rangle \\ &= (I_X)_z((((s,t] \times A) \cap [[0,T]]) \times B)(f). \end{aligned} \tag{3.36}$$

The second equality in (3.36) is by (ii) of (a). It follows easily that

$$|(I_{X^T})_z|((s,t] \times A \times B) = |(I_X)_z|(((s,t] \times A) \cap [[0,T]]) \times B) \tag{3.37}$$

The equality in (3.37) is valid on $\mathcal{R}$ (as the variation of a countably (finitely) additive measure is countably (finitely) additive (see Dinculeanu [12], page 35). Observe that since X is summable we have $|(I_X)_z|(((0,\infty] \times \Omega) \times \mathbb{R}) < \infty$ (we are using the fact that I_X has bounded semivariation on $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ and proposition (2.3.13) which says, in our case, that $(\tilde{I}_X)_{F,L^p_G} = \sup\{|(I_X)_z| : z \in (L^q_{G^*})_1\}$. Hence $(I_X)_z$ has finite variation). Similarly since X^T is summable $(I_{X^T})_z$ also has finite variation.

Step 2. If H is a simple predictable process of the form

$$H = \sum_{i=1}^{n} 1_{(s_i,t_i] \times A_i} 1_{B_i} f_i \tag{3.38}$$

where $A_i \in \mathcal{F}_{s_i}$, $B_i \in \mathcal{B}(\mathbb{R})$ and $f_i \in F$, we have

$$\begin{aligned}\int |H| \; d|(I_{X^T})_z| &= \sum_{i=1}^{n} |f_i||(I_{X^T})_z|((s_i, t_i] \times A_i \times B_i) \\ &= \sum_{i=1}^{n} |f_i||(I_X)_z|[(((s_i, t_i] \times A_i) \cap [[0, T]]) \times B] \\ &= \int 1_{[[0,T]]}|H| \; d|(I_X)_z|.\end{aligned} \tag{3.39}$$

The second equality in (3.39) is by (3.37). Thus far we have established (3.35) for simple predictable functions H.

Step 3. Now let $H \in \mathcal{F}_{F,G}(X)$. Since H is predictable we can obtain a sequence $(H_n)_{n=1}^{\infty}$ of simple predictable functions such that $H_n \longrightarrow H$ pointwise and $|H_n| \leq |H|$. By monotone convergence we have

$$\int |H_n| \; d|(I_{X^T})_z| \longrightarrow \int |H| \; d|(I_{X^T})_z| \tag{3.40}$$

and also

$$\int 1_{[[0,T]]}|H_n| \; d|(I_X)_z| \longrightarrow \int 1_{[[0,T]]}|H| \; d|(I_X)_z|. \tag{3.41}$$

Since $\int |H_n| \; d|(I_{X^T})_z| = \int 1_{[[0,T]]}|H_n| \; d|(I_X)_z|$ by step (2), we arrive at the equality

$$\int |H| \; d|(I_{X^T})_z| = \int 1_{[[0,T]]}|H| \; d|(I_X)_z|.$$

This proves the equality in (3.35) and thus also the theorem. ∎

We are now in a position to prove the main theorem of this section. It says that if T is a stopping time, $c_0 \not\subset G$ and $H \in L^1_{F,G}(X)$ then the summable process measure $(H \cdot X)^T$ is equal to $H \cdot X^T$.

THEOREM (3.4.6). Assume

(a) $H \in L^1_{F,G}(X)$

and

(b) $\int_D H \; dI_X \in L^p_G$ for every $D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$.

Let T be a stopping time. We have

(1) $1_{[[0,T]]}H \in L_{F,L^p_G}(X)$ and $H \in L^1_{F,G}(X^T)$.

(2) For any $B \in \mathcal{B}(\mathbb{R})$ and $t \in [0,\infty]$ we have

$$(H \cdot X)^T_t(B) = ((1_{[[0,T]]}H) \cdot X)_t(B) = (H \cdot X^T)_t(B).$$

Further, if $c_0 \not\subset G$ then we have

(3) (a) $(H \cdot X)$, $(H \cdot X)^T$ and $(H \cdot X^T)$ are summable relative to $(\mathbb{R}, L^p_G)$.

(b) The process measures $(H \cdot X)^T$ and $(H \cdot X^T)$ are equal, $(H \cdot X)^T = H \cdot X^T$.

Proof. The proof for the first part of (1) and the first equality in (2) is done the same way as the proof of theorem 3.8 in Brooks and Dinculeanu [4]. We sketch that proof here.

Consider first a simple stopping time T of the form $T = \sum_{1 \le i \le n} 1_{A_i} t_i$ with $0 \le t_1 \le \cdots < t_n \le +\infty$ and $A_i \in \mathcal{F}_{t_i}$, where the A_i form a partition of Ω. We have

$$(H \cdot X)_T(B)(w) = (H \cdot X)_{t_i}(B)(w) = \left(\int_{[0,t_i]\times B} H \, dI_X\right)(w), \tag{3.42}$$

where t_i is such that $T(w) = t_i$. Thus,

$$\begin{aligned}
(H \cdot X)_T(B) &= \sum_{1 \le i \le n} 1_{A_i} \int_{[0,t_i]\times B} H \, dI_X \\
&= \int_{[0,\infty]\times\Omega\times B} H \, dI_X - \sum_{1 \le i \le n} 1_{A_i} \int_{(t_i,\infty]\times B} H \, dI_X \\
&= \int_{[0,\infty]\times\Omega\times B} H \, dI_X - \sum_{1 \le i \le n} 1_{A_i} \int_{(t_i,\infty]\times B} H \, dI_X \\
&= \int_{[0,\infty]\times\Omega\times B} H \, dI_X - \int \sum_{1 \le i \le n} (1_{(t_i,\infty]} 1_{A_i} 1_B H) \, dI_X \\
&= \int_{[0,\infty]\times\Omega\times B} H \, dI_X - \int 1_{((T,\infty]]} 1_B H \, dI_X.
\end{aligned} \tag{3.43}$$

The fourth equality in (3.43) is by Proposition 3.4.2 (condition I). The fifth equality is obtained because $((T,\infty]] = \bigcup_{i=1}^n (t_i,\infty] \times A_i$ and thus $1_{((T,\infty]]} = \sum_{i=1}^n 1_{(t_i,\infty]} 1_{A_i}$. It follows from (3.4.3) that

$$(H \cdot X)_T(B) = \int_{[0,\infty]\times\Omega\times B} H \, dI_X - \int_{((0,T]]\times B} H \, dI_X = \int 1_{[[0,T]]} 1_B H \, dI_X. \tag{3.44}$$

Now suppose that T is an arbitrary stopping time. Then we may approximate T by a sequence (T_n) of simple stopping times from above. An application of the Lebesgue theorem (theorem (3.1.1)) allows us to conclude that (3.44) is valid for the stopping time T. Replacing T by $T \wedge t$ in (3.44) we obtain:

$$(H \cdot X)_t^T(B) = \int_{[0,t]\times B} 1_{[[0,T]]} H \, dI_X. \tag{3.45}$$

Observe that assumptions (a) and (b) imply (via theorem (3.4.3)) that $H \cdot X$ is p-summable relative to $(\mathbb{R}, L_G^p)$ and, hence from proposition (3.3.1) we conclude that $(H \cdot X)^T$ is a process measure. This fact and (3.45) imply that $(1_{[[0,T]]}H) \in L^1_{F,G}(X)$. It follows that

$$(H \cdot X)_t^T(B) = ((1_{[[0,T]]}H) \cdot X)_t(B).$$

This gives the first part of (1) and the first equality in (2).

We now prove the second part of (1) and the second equality in (2). Infact from theorem (3.4.5 (b)) we know that $(\tilde{I}_{X^T})_{F,L_G^p}(X)$ is finite so that $H \in \mathcal{F}_{F,G}(X^T)$. Hence if we prove the second equality in (2) we will also have that $H \in L^1_{F,G}(X^T)$. We do this next. We need to verify that

$$\int 1_{[0,t]} 1_B H \, dI_{X^T} = \int 1_{[0,t]} 1_B H 1_{[[0,T]]} dI_X \tag{3.46}$$

Returning to section 3 of chapter 2 we realize that since the integral is an element in $(L^q_{G^*})^*$ it suffices to show that

$$\left\langle \int 1_{[0,t]} 1_B H \, dI_{X^T}, z \right\rangle = \left\langle \int 1_{[0,t]} 1_B 1_{[[0,T]]} H \, dI_X, z \right\rangle \tag{3.47}$$

for each $z \in L^q_{G^*}$.

If H is a simple predictable process of the form

$$H = \sum_{1 \le i \le n} 1_{(s_i,t_i]} 1_{A_i} 1_{B_i} f_i, \quad A_i \in \mathcal{F}_{s_i}, \; f_i \in F \tag{3.48}$$

then we have

$$
\begin{aligned}
\left\langle \int_{[0,t]\times B} H \, dI_{X^T}, z \right\rangle &= \int_{[0,t]\times B} \sum_{i=1}^{n} 1_{A_i} 1_{B_i} f_i \, d(I_{X^T})_z \\
&= \sum_{i=1}^{n} (I_{X^T})_z(([0,t] \times A_i) \times (B_i \cap B)) f_i \\
&= \left\langle \sum_{i=1}^{n} (I_{X^T})(([0,t] \times A_i) \times (B_i \cap B)) f_i, z \right\rangle \\
&= \left\langle \sum_{i=1}^{n} (I_X)((([0,t] \times A_i) \cap [[0,T]]) \times B_i \cap B) f_i, z \right\rangle \qquad (3.49) \\
&= \sum_{i=1}^{n} (I_X)_z((([0,t] \times A_i) \cap [0,T]) \times B_i \cap B) f_i \\
&\ \ \vdots \\
&= \int_{[0,t]\times B} \sum_{i=1}^{n} 1_{[[0,T]]} 1_{A_i} 1_{B_i} \, d(I_X)_z \\
&= \left\langle \int_{[0,t]\times B} 1_{[[0,T]]} H \, dI_X, z \right\rangle.
\end{aligned}
$$

The fourth equality in (3.49) is obtained via part (a) (ii) of theorem (3.4.5). Thus (3.47) is true for any simple predictable H of the form in (3.48). Now let ($H \in L^1_{F,G}(X)$). Then $H \in \mathcal{F}_{F,G}(X)$ and thus also to $\mathcal{F}_{F,G}(X^T)$ (theorem 3.4.5 (b)). Let $(H_n)_{n=1}^{\infty}$ be a sequence of simple predictable functions such that $H_n \longrightarrow H$ pointwise and $|H_n| \leq |H|$. Then $H_n, H \in L^1(|(I_{X^T})_z|)$. Using the equality in (3.47) which is true for simple predictable functions and the dominated convergence theorem we conclude that (3.47) remains true for $H \in L^1_{F,G}(X)$. This completes the proof of statements (1) and (2) of the theorem. Finally statement (3) follows directly from statements (1) and (2), theorem (3.4.3) and the theorem (3.4.5), part (a). ∎

3.5 Some Properties of the Space $L^1_{F,G}(X)$

The purpose of this section to bring to the readers' notice that the space $L^1_{F,G}(X)$ possesses some of the familiar properties that one encounters in the standard theory of stochastic integration [4, 10, 27]. We remark at this stage that though we were tempted to consider two parameter stopping times, we did not do so. This is because

such stopping times have many shortcomings (see [31]).

The positive results that we have concerning the space $L^1_{F,G}(X)$ are (1) $L^1_{F,G}(X)$ is complete and (2) $L^1_{F,G}(X)$ contains the bounded σ-elementary processes (to be described shortly). Since the proofs are carried out in a similar fashion to the proofs of the corresponding results [4], we find little need to reproduce them here. We will direct the reader to the specific theorems [4] at the appropriate time. There are, however, certain details that must be added and certain remarks that must be made. These embellishments will then be the primary worth of this section. The proof of the next theorem (on completeness of $L^1_{F,G}(X)$) is the same as that for theorem (3.10) in Brooks and Dinculeanu [4].

THEOREM (3.5.1). Let $(H^n)_{n=1}^{\infty}$ be a sequence in $L^1_{F,G}(X)$. Then there is an $H \in \mathcal{F}_{F,G}(X)$ such that $H^n \longrightarrow H$ in the seminorm $(\tilde{I}_X)_{F,L^p_G}$. For each $t \in [0,\infty]$, $B \in \mathcal{B}(\mathbb{R})$, we have

(a) $(H^n \cdot X)_t(B) \longrightarrow (H \cdot X)_t(B)$ in L^p_G and

(b) There exists a subsequence (n_k) such that $(H^{n_k} \cdot X)_t(B) \longrightarrow (H \cdot X)_t(B)$ a.s. as $k \longrightarrow \infty$, uniformly on every compact time interval.

(c) $H \in L^1_{F,G}(X)$.

REMARK (3.5.2).

(a) The existence of H in the theorem is immediate as the space $\mathcal{F}_{F,G}(X)$ is complete (see remark (2.3.33)).

(b) We remind the reader that the elements $(H^n)_{n=1}^{\infty}$ of $\mathcal{F}_{F,G}(X)$ are predictable (see step (6) in section 1 of chapter 3). It follows that H is also predictable.

(c) The proof of this theorem involves stopping times in an essential way. These stopping times are associated with the family $((H^n \cdot X)(B)(\cdot))_{n=1}^{\infty}$. To be more

precise the family $(T_n)_{n=1}^{\infty}$ of stopping times used in the proof are defined as follows:

$$T_n = \inf\left\{t : \left|(H^n \cdot X)_t(B) - (H^{n+1} \cdot X)_t(B)\right| > \frac{1}{2^n}\right\} \wedge t_0 \tag{3.50}$$

where t_0 is some positive fixed number.

(d) In order to show that $H \in L^1_{F,G}(X)$ we must verify the two conditions in definition (3.2.2). Condition (2) obtains immediately from the proof of theorem (3.10) in Brooks and Dinculeanu [4], mentioned earlier; we omit the details. It remains for us to verify condition (1). From (a) of the theorem we know that the set function $(H \cdot X)_t(\cdot) : \mathcal{B}(\mathbb{R}) \longrightarrow L^p_G$ exists and is given by $(H \cdot X)_t(A) = \lim_{n \longrightarrow \infty}(H^n \cdot X)_t(A)$, where the limit is in L^p_G. Since $H^n \in L^1_{F,G}(X)$, each of the set functions $(H^n \cdot X)_t(\cdot)$. $\mathcal{B}(\mathbb{R}) \longrightarrow L^p_G$ is a σ-additive measure. Thus by the generalized Nikodym theorem (theorem 6, page 321 in [18]), we have that $(H \cdot X)_t(\cdot)$ is also a L^p_G valued σ-additive measure. Thus $H \in L^1_{F,G}(X)$.

We now proceed to our second positive result.

Definition (3.5.3). We call the F-valued process $H : \mathbb{R}_+ \times \Omega \times \mathbb{R} \longrightarrow F$ σ-elementary if H is of the form

$$H = H_0 1_{\{0\}} 1_{B_0} + \sum_{1 \leq n < \infty} H_n 1_{((T_n, T_{n+1}]]} 1_{B_n} \tag{3.51}$$

where $(H_n)_{n \geq 0}$ is a bounded family of random variables such that $H_0 \in \mathcal{F}_0$, $H_n \in \mathcal{F}_{T_n}$, $n = 1, 2, \ldots$, the sets $(B_n)_{n \geq 0}$ are in $\mathcal{B}(\mathbb{R})$ and $(T_n)_{n=1}^{\infty}$ is a sequence of stopping times with $T_n \uparrow \infty$.

The next result is of importance because it will allow us to prove that the σ-elementary processes in $\mathcal{F}_{F,G}(X)$ are in $L^1_{F,G}(X)$. A proof of this theorem may be adapted from the proof of theorem 3.16 in [4].

Theorem (3.5.4). Let $H \in \mathcal{F}_{F,G}(X)$. Suppose that the following condition is satisfied:

There exists a family $(T_n)_{n=1}^{\infty}$ of stopping times with $T_n \uparrow \infty$ and for each n we have $(1_{[[0,T_n]]}H) \in L^1_{F,G}(X^{T_n})$. (3.52)

Then $H \in L^1_{F,G}(X)$ and for each $B \in \mathcal{B}(\mathbb{R})$ we have

$$(H \cdot X)_t(B) = \lim_{n \longrightarrow \infty}((1_{[[0,T_n]]}H) \cdot X)_t(B) \text{ pointwise for each } t \geq 0. \tag{3.53}$$

REMARK (3.5.5). The proof of this theorem involves arguments that are similar to the proof of theorem 3.16 [4]. A close inspection of the proof of this theorem yields two facts. For each $B \in \mathcal{B}(\mathbb{R})$ and any $t \geq 0$ we have

(a) $(\int_{[0,t]\times B} H\ dI_X) \in L^p_G$ and

(b) $\int_{[0,t]\times B} 1_{[[0,T_n]]}H\ dI_X \longrightarrow \int_{[0,t]\times B} H\ dI_X$ in L^p_G.

In order to show that $H \in L^1_{F,G}(X)$ we need to verify condition (1) of definition (3.2.2) (note that $(H \cdot X)_t(B)$ is cadlag, adapted, by virtue of (3.53)). From fact (b) we have that

$$\lim_{n \longrightarrow \infty}((1_{[[0,T_n]]}H) \cdot X)_t(B) = \int_{[0,t]\times B} H\ dI_X \tag{3.54}$$

for each $B \in \mathcal{B}(\mathbb{R})$. The limit in (3.54) is in L^p_G. Observe that for each n the set function $(1_{[[0,T_n]]}H \cdot X)_t(\cdot) : \mathcal{B}(\mathbb{R}) \longrightarrow L^p_G$ is a σ-additive measure. Hence by the generalized Nikodym theorem (18, theorem 6, page 321) we conclude that the set function $\int_{[0,t]\times B} H\ dI_X : \mathcal{B}(\mathbb{R}_+) \longrightarrow L^p_G$ is also a σ-additive measure. Hence, indeed, $H \in L^1_{F,G}(X)$.

We are now in a position to prove our next result.

THEOREM (3.5.6). Let $H \in \mathcal{F}_{F,G}(X)$ be a σ-elementary process of the form in (3.51). Then $H \in L^1_{F,G}(X)$. For any $B \in \mathcal{B}(\mathbb{R})$ we may write $(H \cdot X)_t(B)$ as

$$(H \cdot X)_t(B) = H_0X_0(B_0 \cap B) + \sum_{1 \leq n < \infty} H_n[X_{T_{n+1}\wedge t}(B_n \cap B) - X_{T_n \wedge t}(B_n \cap B)] \tag{3.55}$$

Proof. For each n put

$$\begin{aligned} G_n &= 1_{[[0,T_n]]}H \\ &= 1_{[[0,T_n]]}\left(H_0 1_{\{0\}}1_{B_0} + \sum_{1\le k<\infty} H_k 1_{((T_k,T_{k+1}]]}1_{B_k}\right) \\ &= H_0 1_{\{0\}}1_{B_0} + \sum_{k=1}^{n-1} H_k 1_{((T_k,T_{k+1}]]}1_{B_k}. \end{aligned}$$

It follows that for each n, G_n is bounded, predictable and $\widetilde{I}_X(G_n) < \infty$ (note: X is p-summable). Hence, for each n, $G_n \in \mathcal{F}_{F,G}(X)$. Observe that for each n, G_n is an elementary process (see (3.6)) and hence by corollary (3.3.3), $G_n \in L^1_{F,G}(X)$. Using just the fact that $H \in \mathcal{F}_{F,G}(X)$, we obtain via theorem (3.4.5 (b)) that

$$(\widetilde{I}_{X^{T_n}})_{F,L^p_G}(H) = (\widetilde{I}_X)_{F,L^p_G}(1_{[[0,T_n]]}H) < \infty. \tag{3.56}$$

It is immediate from (3.56) that $(\widetilde{I}_{X^{T_n}})_{F,L^p_G}(1_{[[0,T_n]]}H)$ is finite. Next since $(1_{[[0,T_n]]}H) \in L^1_{F,G}(X)$ we have by part (2) of theorem (3.4.6) that

$$((1_{[[0,T_n]]}H)\cdot X^{T_n})_t(B) = ((1_{[[0,T_n]]}H)\cdot X)_t(B). \tag{3.57}$$

Note that in establishing (3.57) we made no use of assumption (b) in theorem (3.4.6); a close inspection of the proof of that theorem will reveal the truth of this statement. From (3.56) and (3.57) we conclude that $(1_{[[0,T_n]]}H) \in L^1_{F,G}(X^{T_n})$. Thus condition (3.52) of theorem (3.5.4) is satisfied. It follows from theorem (3.5.4) and remark (3.5.5) that $H \in L^1_{F,G}(X)$. It remains for us to obtain (3.55). Using Corollary (3.3.3) we can write $(G_n \cdot X)_t(B)$ as

$$(G_n\cdot X)_t(B) = H_0X_0(B\cap B_0) + \sum_{1\le i\le n-1} H_i[X_t^{T_{i+1}}(B\cap B_i) - X_t^{T_i}(B\cap B_i)].$$

Finally using (3.53) $(H\cdot X)_t(B)$ may be written as

$$\begin{aligned} (H\cdot X)_t(B) &= \lim_{n\longrightarrow\infty}(G_n\cdot X)_t(B) \quad \text{(the limit is pointwise)} \\ &= H_0X_0(B_0\cap B) + \sum_{1\le n<\infty} H_n[X_{T_{n+1}\wedge t}(B_n\cap B) - X_{T_n\wedge t}(B_n\cap B)]. \end{aligned}$$

This gives (3.54). The theorem is proved. ■

3.6 The Stochastic Integral $(H \cdot X)$ When the Process Measure X Is a Martingale

The standing assumption on X throughout this chapter has been that it is a p-summable process measure. In this section we suppose further that it is a martingale. That is, for each $B \in \mathcal{B}(\mathbb{R})$ fixed we assume that $X(B)$ is a martingale. Now suppose that $H \in \mathcal{F}_{F,G}(X)$. The question that we seek to answer here is the following: under what conditions will $(H \cdot X)$ turn out to be a p-summable (relative to an appropriate pair of spaces) process measure that is also a martingale?

As we will see in the theorem that we state we require that $H \in L^1_{F,G}(X)$. The reason for such a stringent requirement is that we will need to use the adaptedness of $(H \cdot X)(B)$ for any $B \in \mathcal{B}(\mathbb{R})$, in an essential way to show that $(H \cdot X)$ is a martingale. The reader is directed to remark (3.6.2) (and Corollary (3.6.3)) for alternate assumptions that also yield the conclusion of the theorem. The reader should realize by now the fundamental theme of this chapter. The theme is, of course, that the stochastic integral enjoys the same characteristics as X. Thus if X is a p-summable process measure then so is $H \cdot X$ (under certain conditions; theorem (3.4.3)). In the next theorem we see that if X is also a martingale then the same is true of $H \cdot X$. Finally in section 3 of chapter 4 we will discover that if X is an orthogonal martingale measure then so is $H \cdot X$ (theorem (4.3.7).

THEOREM (3.6.1). Let X be a process measure that is p-summable relative to (F, G) and let $H \in L^1_{F,G}(X)$. Then the family $((H \cdot X)(B))_{t \geq 0, B \in \mathcal{B}(\mathbb{R})}$ is bounded in L^p_G.

If X is a martingale then $(H \cdot X)$ is a martingale. Further, if $(\int_D H \, dI_X) \in L^p_G$ for every $D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ (which is true if G is reflexive) then $(H \cdot X)$ is p-summable relative to $(\mathbb{R}, L^p_G)$.

Proof. Computing $||(H\cdot X)_t(B)||_{L^p_G}$ we have

$$
\begin{aligned}
||(H\cdot X)_t(B)||_{L^p_G} &= \sup_{z\in(L^q_{G^*})_1}\left|\left\langle\int_{[0,t]\times B} H\,dI_X, z\right\rangle\right| \\
&\le \sup_{z\in(L^q_{G^*})}\int |H|\,d|(I_X)_z| \qquad (3.58)\\
&\le (\tilde{I}_X)_{F,L^p_G}(H)<\infty.
\end{aligned}
$$

The last inequality in (3.58) is true since $H\in L^1_{F,G}(X)$. It follows from (3.58) that:

$$\sup_{B\in\mathcal{B}(\mathbb{R})}\sup_{t\in\mathbb{R}_+}||(H\cdot X)_t(B)||_{L^p_G}<\infty. \qquad (3.59)$$

Thus the family $((H\cdot X)_t(B))_{t\ge 0,B\in\mathcal{B}(\mathbb{R})}$ is bounded in L^p_G.

The last statement of the theorem follows at once from theorem (3.4.3).

It remains for us to prove that for any $B\in\mathcal{B}(\mathbb{R})$ fixed, $(H\cdot X)(B)$ is a martingale. Fix such a $B\in\mathcal{B}(\mathbb{R})$. It suffices for us to show that for any $t\ge 0$ the following holds:

$$E[1_C\int_{(t,\infty]\times B} H\,dI_X|\mathcal{F}_t]=0 \text{ for every } C\in\mathcal{F}_t. \qquad (3.60)$$

Now suppose (3.60) holds and let $0\le s\le t$. Then we have

$$
\begin{aligned}
&E[(H\cdot X)_t(B)-(H\cdot X)_s(B)|\mathcal{F}_s]\\
&=E\left[\int_{(s,\infty]\times B} H\,dI_X-\int_{(t,\infty]\times B} H\,dI_X\middle|\mathcal{F}_s\right] \qquad (3.61)\\
&=E\left[\int_{(s,\infty]\times B} H\,dI_X|\mathcal{F}_s\right]-E\left[E\left[\int_{(t,\infty]\times B} H\,dI_X\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_s\right]=0
\end{aligned}
$$

The first step in (3.61) uses the fact that $(H\cdot X)_t(B)\in\mathcal{F}_t$ for every t. Thus (3.61) shows that $(H\cdot X)(B)$ is a martingale.

We now prove (3.60). Assume first that H is a simple predictable process of the form:

$$H=1_{\{0\}}1_A 1_B x, \quad \text{where } A\in\mathcal{F}_0 \text{ and } B\in\mathcal{B}(\mathbb{R}) \qquad (3.62)$$

and let $t\ge 0$ be fixed. In this case it is clear that (3.60) holds for H of the form in (3.62). Next suppose H has the form

$$H=1_{(r,s]}1_A 1_B x, \quad \text{where } A\in\mathcal{F}_r \text{ and } B\in\mathcal{B}(\mathbb{R}). \qquad (3.63)$$

If $s \le t \vee r$ then (3.60) is obviously satisfied for this H. Hence assume that $s > t \vee r$. Then for any $B' \in \mathcal{B}(\mathbb{R})$ we have

$$\begin{aligned} \int_{(t,\infty]\times B'} H\, dI_X &= x \int_{(t,\infty]\times B'} 1_{(r,s]} 1_A 1_B\, dI_X \\ &= x \int 1_{(t\vee r,s]} 1_A 1_{B\cap B'} dI_X \\ &= x 1_A [X_s(B\cap B') - X_{t\vee r}(B\cap B')] \end{aligned} \tag{3.64}$$

From (3.64) it follows that

$$1_C \int_{(t,\infty]} H\, dI_X = X 1_{C\cap A}[X_s(B\cap B') - X_{t\vee r}(B\cap B')] \tag{3.65}$$

where $C \in \mathcal{F}_t$. Taking expectations in (3.65) we have

$$\begin{aligned} E\left[1_C \int_{(t,\infty]} H\, dI_X\right] &= xE[1_{C\cap A}(X_s(B\cap B') - X_{t\vee r}(B\cap B'))] \\ &= xE[1_{C\cap A} E[X_s(B\cap B') - X_{t\vee r}(B\cap B')|\mathcal{F}_{t\vee r}]] \\ &= xE[1_{C\cap A}[X_{t\vee r}(B\cap B') - X_{t\vee r}(B\cap B')]] = 0. \end{aligned} \tag{3.66}$$

For the computations in (3.66) we have used the following three facts: (1) $s > t \vee r$ (2) $C \cap A \in \mathcal{F}_{t\vee r}$ and (3) $X(B)$ is a martingale. From the definition of conditional expectation, (3.60) holds true for H of the form in (3.63). It follows that (3.60) remains true for $\mathcal{R}$-measurable simple functions $H(\mathcal{R} =$ ring generating $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}))$.

Now let $H \in L^1_{F,G}(X)$ be arbitrary. The remainder of the proof is the same as the proof of theorem 3.21 [4]. We present it here for completeness. Observe that $E[1_C \int_{(t,\infty]} H\, dI_X]$ is G-valued. Hence in order to show (3.60) it is enough if we show that

$$\left\langle E\left[1_C \int H\, dI_X\right], g^*\right\rangle = 0 \text{ for each } g^* \in G^*. \tag{3.67}$$

Since $H \in L^1_{F,G}(X)$ the quantity $(\tilde{I}_X)_{F,G}(H)$ is finite. Then, using proposition (2.3.30) we have that $H \in L^1_F((I_X)_z)$ for $z \in L^q_{G^*}$. In particular, we may take z to be $z = 1_C g^* \in L^q_{G^*}$, where $g^* \in G^*$. Let (H^n) be a sequence of $\mathcal{R}$-measurable simple functions with $H^n \longrightarrow H$ in $L^1_F((I_X)_z)$. We have $\langle \int_{(t,\infty]\times B} H^n\, dI_X, z\rangle \longrightarrow \langle \int_{(t,\infty]\times B} H\, dI_X, z\rangle$.

Using the duality between L^p_G and $L^q_{G^*}$ (i.e., $\langle J, K\rangle = E[\langle J(w), K(w)\rangle]$, where $J \in L^p_G$ and $K \in L^q_{G^*}$) and replacing z by $1_C g^*$ we have

$$E\left[\left\langle 1_C \int_{(t,\infty]\times B} H^n \, dI_X, g^*\right\rangle\right] \longrightarrow E\left[\left\langle 1_C \int_{(t,\infty]\times B} H \, dI_X, g^*\right\rangle\right]$$

or

$$\left\langle E\left[1_C \int_{(t,\infty]\times B} H^n \, dI_X\right], g^*\right\rangle \longrightarrow \left\langle E\left[1_C \int_{(t,\infty]\times B} H \, dI_X\right], g^*\right\rangle. \tag{3.68}$$

Since the left hand side of (3.68) is zero for all n and as $g^* \in G^*$ is arbitrary we conclude that $E[1_C \int_{(t,\infty]} H \, dI_X] = 0$ for any $C \in \mathcal{F}_t$. This gives (3.60) and the theorem is proved. ■

REMARK (3.6.2). Instead of the assumption that $H \in L^1_{F,G}(X)$ in the theorem we may assume the following: (1) $H \in \mathcal{F}_{F,G}(X)$, (2) $\int_{[0,t]\times B} H \, dI_X$ is $\mathcal{F}_t$ adapted for every $t \geq 0$ and for each $B \in \mathcal{B}(\mathbb{R})$ and (3) $\int_D H \, dI_X$ is L^p_G valued for every $D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$. Condition (3) implies via theorem (2.3.42) that $d(HI_X)$ is a σ-additive measure on $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$. One may then prove, as above, that $(\int_{[0,t]\times B} H \, dI_X)_{t\geq 0}$ is a martingale. Finally it is a well known fact that every (Banach valued) martingale has a cadlag modification (see [10], chapter VI for the real valued case and the reference in [4] for the Banach valued case). Hence $H \in L^1_{F,G}(X)$. The next corollary says that we may remove the stringent requirement that $H \in L^1_{F,G}(X)$ in theorem (3.6.1) and instead replace it with the condition $H \in \mathcal{F}_{F,G}(\mathcal{B}, X)$ (the closure of the bounded predictable functions in $\mathcal{F}_{F,G}(X)$), where G does not contain a copy of c_0.

COROLLARY (3.6.3). Assume that $c_0 \not\subset G$. Let X be a process measure that is p-summable relative to (F, G) and let $H \in \mathcal{F}_{F,G}(\mathcal{B}, X)$. Then

(a) The family $((H \cdot X)_t(B))_{t\geq 0, B\in\mathcal{B}(\mathbb{R})}$ is bounded in L^p_G.

(b) If X is a martingale then $(H \cdot X)$ is a martingale and $H \in L^1_{F,G}(\mathcal{B}, X)$.

(c) $(H \cdot X)$ is p-summable relative to $(\mathbb{R}, L^p_G)$.

Proof. Since $c_0 \not\subset G$ we may invoke theorem (2.3.20) to conclude that the family of measures $(I_X)_{F,G}$ is uniformly σ-additive.

We claim that for each $D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ we have $\int_D H\, dI_X \in L^p_G$. Once the claim is proved we may conclude from theorem (2.3.42) (a) that the set function

$$\int_{(\cdot)} H\, dI_X : \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}) \longrightarrow L^p_G(\Omega, \mathcal{F}, P)$$

σ-additive measure. In particular, for any $t \geq 0$ fixed the set function $\int_{[0,t]\times(\cdot)} H\, dI_X : \mathcal{B}(\mathbb{R}) \longrightarrow L^p_G$ is a σ-additive measure. This means that H satisfies condition (1) in definition (3.2.2).

Observe also that for each $B \in \mathcal{B}(\mathbb{R})$ and for any $t \geq 0$, $\int_{[0,t]\times B} H\, dI_X$ is $\mathcal{F}_t$-adapted by theorem (3.2.1). The cadlag requirement in condition (2) will be exhibited soon. It then follows that $H \in L^1_{F,G}(\mathcal{B}, M)$.

We now prove the claim made earlier. Let $D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$. Observe first that since the family of measures $(I_X)_{F,G}$ is uniformly σ-additive we have via proposition (2.3.39) the following equalities:

$$\mathcal{F}_{F,G}(S(\mathcal{R}), I_X) = \mathcal{F}_{F,G}(S(\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})), I_X) = \mathcal{F}_{F,G}(\mathcal{B}, I_X).$$

Thus let $(H^n)_{n=1}^\infty$ be a sequence of $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ simple functions such that

$$\left\| \int_D H^n\, dI_X - \int_D H\, dI_X \right\|_{L^p_G} \leq (\tilde{I}_X)_{F,G}(H^n - H) \longrightarrow 0.$$

For each n we have $\int_D H^n\, dI_X \in L^p_G$, hence it follows that $\int_D H\, dI_X \in L^p_G$. This proves the claim. We can now prove statements (a), (b) and (c) quickly. Statement (a) is done the same way as in theorem (3.6.1). The same proof as in theorem (3.6.1) yields that $(H \cdot X)$ is a martingale (note: to prove this statement we only used the fact that for each $B \in \mathcal{B}(\mathbb{R})$, the process $(\int_{[0,t]\times B} H\, dI_X)_{t\geq 0}$ is $\mathcal{F}_t$-adapted and that $H \in \mathcal{F}_{F,G}(X)$ there). By remark (3.6.2) we know that $(\int_{[0,t]\times B} H\, dI_X)_{t\geq 0}$ has a cadlag modification. Hence $H \in L^1_{F,G}(\mathcal{B}, X)$ (see our earlier discussion where we checked to see which conditions of definition (3.2.2) the process H satisfied). This proves statement (b). Finally statement (c) follows by virtue of theorem (3.4.3) as $\int_D H\, dI_X \in L^p_G$ for every $D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$. ∎

CHAPTER 4
SUMMABLE PROCESS MEASURES

In chapter 3 we presented a theory of the stochastic integral with respect to a p-summable process measure. This theory will be useful only insofar as there exist p-summable process measures. The purpose of this chapter is to prove that certain types of process measures are p-summable. The main result of this chapter is that orthogonal martingale measures, under certain restrictions, are 2-summable (see theorem (4.2.8)). We also prove in section 4 that a special class of martingale measures with nuclear covariance is 2-summable.

To set matters in their proper perspective we return briefly to bimeasures. From our work in chapter 1, (lemma (1.3.3)), we know that a bimeasure $\beta : \Sigma_1 \times \Sigma_2 \longrightarrow E$ is finitely additive on $\sum_1 \times \sum_2$ (the σ-algebras Σ_1 and Σ_2 will now in our context be $\mathcal{P}_\infty$ and $\mathcal{B}(\mathbb{R})$ respectively. In particular for a process measure (chapter 2, section 2) $I_X : \mathcal{P}_\infty \times \mathcal{B}(\mathbb{R}) \longrightarrow L^p_E$ is finitely additive on $\mathcal{P}_\infty \times \mathcal{B}(\mathbb{R})$. We show that in the case when $p = 2$, E is any Hilbert space, and X is one of the special process measures in the above paragraph, then I_X can be extended to a σ-additive measure on $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ with finite semivariation. This chapter is partitioned as follows: In section 1 we gather all the preliminary notions that will be required for section 2. In section 2 we show that if M is an orthogonal martingale measure and if $\sup_{B \in \mathcal{B}(\mathbb{R})} \langle M(B) \rangle_\infty$ is finite then M and $\langle M \rangle$ are <u>2-summable</u>.

In section 3 we consider the stochastic integral $(H \cdot M)$ when M is an orthogonal martingale measure. The Ito isometry and the representation of the stochastic integral as an orthogonal martingale measure are demonstrated here.

In section 4 we prove that when M is a martingale measure with nuclear covariance

of the form $M_t(B)(w) = B_t(w)\int_B f\ d\mu$, where $B \in \mathcal{B}(\mathbb{R})$, μ is a finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $(B_t)_{t\geq 0}$ is standard Brownian motion and $f \in L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$, $f \geq 0$, then M is 2-summable.

We record our debt here to the paper by Walsh [47] which provided important ideas that allowed us to establish the results in section 4.2.

4.1 Distributions in $\mathbb{R}^2$ and Non-Cartesian Product measures

Distributions and measures in $\mathbb{R}^2$.

The discussion in this subsection is taken from Billingsley [2] (page 176, mainly). Our aim here is to recall for the reader an analogue in $\mathbb{R}^2$ of the following widely known theorem in $\mathbb{R}^1$. This theorem is about the correspondence between non decreasing right continuous functions and their associated measures.

THEOREM (4.1.1). If $F : \mathbb{R} \longrightarrow \mathbb{R}$ is a nondecreasing right continuous function, then there exists on $\mathcal{B}(\mathbb{R})$ (Borel σ-algebra of $\mathbb{R}$) a unique measure μ (with finite or infinite values) which satisfies the following condition:

$$\mu(a, b] = F(b) - F(a). \tag{4.1}$$

The measure μ is finite if and only if F is bounded.

There is an analogue of theorem (4.1.1) in $\mathbb{R}^n$, $n \geq 2$, but we will be content with $n = 2$ as it suffices for our needs. Let $(t, u) \in \mathbb{R}^2$. We denote by $S_{(t,u)}$ the set

$$S_{(t,u)} = \{(r, s) \in \mathbb{R}^2 | r \leq t, s \leq u\}. \tag{4.2}$$

Thus $S_{(t,u)}$ consists of points "Southwest" of (t, u). Recall that the Borel σ-field $\mathcal{B}(\mathbb{R}^2)$ is generated by bounded rectangles of the form

$$D = (a_1, b_1] \times (a_2, b_2]. \tag{4.3}$$

We can write the set D in terms of the sets $S_{(t,u)}$ in (4.2) as follows:

$$D = S_{(b_1,b_2)} - [S_{(a_1,b_2)} \cup S_{(b_1,a_2)}]. \tag{4.4}$$

It follows that the class of sets in (4.2) generate $\mathcal{B}(\mathbb{R}^2)$. Now suppose that μ is a finite measure on $\mathcal{B}(\mathbb{R}^2)$. In analogy with the one-dimensional case define a real function $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$F(t,u) = \mu(S_{(t,u)}). \tag{4.5}$$

Then for a set D of the form in (4.3) we easily obtain the equality below:

$$\mu(D) = \Delta_D F, \tag{4.6}$$

where

$$\Delta_D F = F(b_1,b_2) + F(a_1,a_2) - F(b_1,a_2) - F(a_1,b_2). \tag{4.7}$$

The σ-additivity of the measure μ implies the following:
For any pair of sequences $t_n \downarrow t$ and $u_n \downarrow u$ we have

$$F(t_n,u_n) \longrightarrow F(t,u) \tag{4.8}$$

We call any real function F on $\mathbb{R}^2$ <u>right continuous</u> if it satisfies (4.8). We are now ready to state our desired theorem.

THEOREM (4.1.2) (2, page 177). Suppose that the real function F on $\mathbb{R}^2$ is right continuous and satisfies $\Delta_D F \geq 0$ for bounded rectangles D. Then there exists a unique measure μ on $\mathcal{B}(\mathbb{R}^2)$ with values in $\overline{\mathbb{R}}_+$ satisfying (4.6) for bounded rectangles D. The measure μ is finite if and only if F is bounded.

The results in the next subsection will play a major role in proving the existence of the positive σ-additive measure in theorem (4.2.8).

<u>Non-Cartesian Product Measures</u>.

The following lemma is taken from Rao [44] (page 335, exercise 2). The informed reader will be aware that this lemma is the basis for defining 'transition' probabilities in probability theory.

LEMMA (4.1.3). Suppose (Ω_i, Σ_i), $i = 1,2$ are a pair of measurable spaces, and $\mu_1 : \Sigma_1 \longrightarrow \overline{\mathbb{R}}_+$ is a measure. Let $\mu_2 : \Sigma_2 \times \Omega_1 \longrightarrow \overline{\mathbb{R}}_+$ be a mapping such that (i) $\mu_2(\cdot, w_1)$ is a measure on Σ_2, $w_1 \in \Omega_1$ and (ii) $\mu_2(A, \cdot)$ is measurable on Ω_1 relative to

Σ_1, for each $A \in \Sigma_2$. Let $\mathcal{S} = \Sigma_1 \times \Sigma_2$ be the semi-algebra of measurable rectangles on $\Omega_1 \times \Omega_2$ and let $\alpha : \mathcal{S} \longrightarrow \overline{\mathbb{R}}_+$ be defined as

$$\alpha(A \times B) = \int_A \mu_2(B, w_1)\mu_1(dw_1), \quad A \times B \in \mathcal{S}. \tag{4.9}$$

Then $\alpha(\cdot)$ is σ-additive on $\mathcal{S}$.

The gist of the next remark is that the outer measure μ^* generated by α is a σ-additive measure on $\Sigma_1 \otimes \Sigma_2$. The remark provides a rough sketch of this fact. This remark may be omitted by the reader who does not want to be drowned in details.

REMARK (4.1.4). Let $\mu^* : \mathcal{S} \longrightarrow \overline{\mathbb{R}}_+$ denote the set function below.

$$\mu^*(A \times B) = \inf \left\{ \begin{array}{l} \displaystyle\sum_{i=1}^{\infty} \alpha(A_i \times B_i); \;\; A \subset \bigcup_{i=1}^{\infty} A_i, \quad B \subset \bigcup_{i=1}^{\infty} B_i, \\ \qquad\qquad\qquad\qquad\;\; A_i \in \Sigma_1, \qquad B_i \in \Sigma_2 \end{array} \right\}. \tag{4.10}$$

By the theorem of C. Caratheodary and E. Hopf 44, theorem 10, page 41) the set function μ^* is an outer measure. Let $\mathcal{M}_{\mu^*}$ denote the μ^* measurable sets on $\Omega_1 \times \Omega_2$. Since α is countably additive on $\mathcal{S}$, we have: (i) $\mathcal{S} \subset \mathcal{M}_{\mu^*}$ and (ii) $\mu^*|_{\mathcal{S}} = \alpha$ (this is by part (iii) of the theorem just mentioned).

Since μ^* is an outer measure (on the power set of $\Omega_1 \times \Omega_2$) Caratheodary's theorem (44, theorem 9, page 37) implies that (i) $\mathcal{M}_{\mu^*}$ is a σ-algebra and (ii) $\mu^*|_{\mathcal{M}_{\mu^*}}$ is a measure. Next, as $\mathcal{S} \subset \mathcal{M}_{\mu^*}$ we conclude that $\Sigma_1 \otimes \Sigma_2 \subset \mathcal{M}_{\mu^*}$. Thus $\mu^*|_{\Sigma_1 \otimes \Sigma_2}$ is a σ-additive measure and $\mu^*|_{\mathcal{S}} = \alpha$.

LEMMA (4.1.5). Continuing with the same notation in lemma (4.1.3) and remark (4.1.4), put $\mu = \mu^*|_{\Sigma_1 \otimes \Sigma_2}$. Then if $f : (\Omega_1 \times \Omega_2, \Sigma_1 \otimes \Sigma_2) \longrightarrow \overline{\mathbb{R}}$ is measurable and μ-integrable we have by a Fubini-Stone theorem for non-cartesian product measures (see Rao [44], exercise 3, page 335) the following equality.

$$\int_\Omega f d\mu = \int_{\Omega_1} \left[\int_{\Omega_2} f(w_1, w_2)\mu_2(dw_2, w_1) \right] \mu_1 \; (dw_1). \tag{4.11}$$

In particular, for $C \in \Sigma_1 \otimes \Sigma_2$ with $\mu(C) < \infty$ we have

$$\mu(C) = \int_{\Omega_1} \left[\int_{\Omega_2} 1_C(w_1, w_2)\mu_2(dw_2, w_1) \right] \mu_1\ (dw_1). \tag{4.12}$$

4.2 Summability of Orthogonal Martingale Measures

The orthogonal martingale measures (chapter 2, section 1) that we consider in this section will be Hilbert valued. Thus E (see chapter 2, section 1) will now denote a Hilbert space. For the remainder of this chapter we shall assume that our orthogonal martingale measures are zero at zero (that is, $M_0(B) = 0,\ \forall\ B \in \mathcal{B}(\mathbb{R})$). In what follows we will devote our attention only to E-valued orthogonal martingale measures M which satisfy the following condition:

$$\sup_{B \in \mathcal{B}(\mathbb{R})} E\left[\langle M(B)\rangle_\infty\right] < \infty. \tag{4.13}$$

We recall for the reader that for each $B \in \mathcal{B}(\mathbb{R})$ the sharp bracket $\langle M(B)\rangle_\infty$ is finite almost surely. Let us indicate why this is true. We know that the process $(M_t(B))_{t\geq 0}$ is a square integrable martingale and hence is right closed by a square integrable random variable which we call $M_\infty(B)$. That is, for each $t \geq 0$ we have $M_t(B) = E(M_\infty(B)|\mathcal{F}_t)$. Observe that for each $t \geq 0$, the set function $M_t(\cdot) : \mathcal{B}(\mathbb{R}) \longrightarrow L^2_E$ is a σ-additive measure. This, together with the fact that $M_t(B) \longrightarrow M_\infty(B)$ in L^2_E for each $B \in \mathcal{B}(\mathbb{R})$, allows us to conclude [via an application of the generalized Nikodym theorem (theorem 6 [18], page 321) that $M_\infty(\cdot) : \mathcal{B}(\mathbb{R}) \longrightarrow L^2_E$ is a σ-additive measure. Now fast forwarding to the computation in (4.72) we have

$$E[1_A(|M_{t_2}(B)|^2_E - |M_{t_1}(B)|^2_E)] = E[1_A(\langle M(B)\rangle_{t_2} - \langle M(B)\rangle_{t_1})] \tag{4.14}$$

where $A \in \mathcal{F}_{t_1}$. Then, in particular, setting $t_2 = \infty$, $t_1 = 0$ and $A = \Omega$ we conclude that $\langle M(B)\rangle_\infty < \infty$ a.s.

The condition in (4.13) is of importance because it allows us to obtain the positive σ-additive measure in theorem (4.2.8), which, in turn, yields the 2-summability of

orthogonal martingale measures. We remark, however, that this result may also be obtained if, instead of (4.13), we replace the σ-algebra $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ (in theorem (4.2.8)) by $\mathcal{P}_\infty \otimes \mathcal{B}(K)$, where K is any compact subset of $\mathbb{R}$.

The next series of results concerning orthogonal martingale measures are preparatory for the main theorem of this section (theorem (4.2.8))

LEMMA (4.2.1). Let $\{M_t(B), (\mathcal{F}_t)_{t\geq 0},\ B \in \mathcal{B}(\mathbb{R})\}$ be an E-valued orthogonal martingale measure which satisfies (4.13). Then the set function $\mu : \mathcal{B}(\mathbb{R}) \longrightarrow [0, +\infty]$ defined by

$$\mu(B) = E[\langle M(B)\rangle_\infty] \tag{4.15}$$

is a finite σ-additive, positive measure.

Proof. Recall that (chapter 2, section 1) $M_t(\cdot) : \mathcal{B}(\mathbb{R}) \longrightarrow L^2_E$ is a σ-additive measure. Thus for $B_1, B_2 \in \mathcal{B}(\mathbb{R})$, disjoint, we have

$$\begin{aligned} E[(M_t(B_1 \cup B_2))^2] &= ||M_t(B_1 \cup B_2)||^2_{L^2_E} \\ &= ||M_t(B_1) + M_t(B_2)||^2_{L^2_E} \\ &= ||M_t(B_1)||^2_{L^2_E} + ||M_t(B_2)||^2_{L^2_E} \\ &= E[(M_t(B_1))^2] + E[(M_t(B_2))^2]. \end{aligned} \tag{4.16}$$

The third equality in (4.16) follows from the orthogonality of M. Next, since M is a square integrable martingale we have

$$\begin{aligned} E[(M_t(B_1 \cup B_2))^2] &= E[\langle M(B_1 \cup B_2)\rangle_t], \\ E[(M_t(B_1))^2] &= E[\langle M(B_1)\rangle_t], \\ \text{and} & \\ E[(M_t(B_2))^2] &= E[\langle M(B_2)\rangle_t]. \end{aligned} \tag{4.17}$$

Combining (4.16) and (4.17) we obtain

$$E[\langle M(B_1 \cup B_2)\rangle_t] = E[\langle M(B_1)\rangle_t] + E[\langle M(B_2)\rangle_t]. \tag{4.18}$$

Replacing t by ∞ in (4.18) we have

$$\mu(B_1 \cup B_2) = \mu(B_1) + \mu(B_2). \tag{4.19}$$

Hence μ is finitely additive. Now let $(B_n)_{n=1}^{\infty}$ be a sequence of sets in $\mathcal{B}(\mathbb{R})$ with $B_n \downarrow \emptyset$. Then

$$\mu(B_n) = E[\langle M(B_n)\rangle_\infty] = E[(M_\infty(B_n))^2] \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

since $M_\infty(\cdot)$ is an L_E^2 measure. It follows that μ is a countably additive measure. Finally, the condition in (4.13) implies that μ is finite. This completes the proof of the lemma. ∎

LEMMA (4.2.2). Let $\{M_t(B), (\mathcal{F}_t)_{t\geq 0}, B \in \mathcal{B}(\mathbb{R})\}$ be an E-valued orthogonal martingale measure which satisfies (4.13). Let $F : \mathbb{R}_+ \times \Omega \times \mathbb{R} \longrightarrow \mathbb{R}_+$ denote the random function

$$F_t(x)(w) = \langle M(-\infty, x]\rangle_t(w) \tag{4.20}$$

where $t \in \mathbb{R}_+$, $w \in \Omega$ and $x \in \mathbb{R}$. Then for any bounded rectangle D of the form $D = (t_1, t_2] \times (x_1, x_2]$, $0 \leq t_1 \leq t_2 \leq +\infty$; $x_1, x_2 \in \mathbb{R}$ and $x_1 \leq x_2$, we have

$$\Delta_D F(w) \geq 0 \quad w\text{.a.s.} \tag{4.21}$$

Proof. Since for each $t \geq 0$, $M_t(\cdot)$ is a L_E^2 valued σ-additive measure it follows from (4.17) that $\langle M(\cdot)\rangle_t$ is a L_E^1 valued σ-additive measure. Hence for B_1, $B_2 \in \mathcal{B}(\mathbb{R})$, disjoint, we have

$$\langle M(B_1 \cup B_2)\rangle_t = \langle M(B_1) + M(B_2)\rangle_t \quad w\text{.a.s.}, \tag{4.22}$$

the null set depends on B_1, B_2 and t. However, the dependence on t can be removed by invoking the right continuity of the sharp bracket. By the orthogonality of M we obtain

$$\langle M(B_1) + M(B_2)\rangle_t = \langle M(B_1)\rangle_t + \langle M(B_2)\rangle_t \quad w\text{.a.s}, \tag{4.23}$$

the null set depending on B_1, B_2.

Hence, we have

$$\begin{aligned} &\langle M(B_1 \cup B_2)\rangle_t(w) = \langle M(B_1)\rangle_t(w) + \langle M(B_2)\rangle_t(w), \\ &w \notin N_{B_1,B_2}, \quad \text{where} \quad P(N_{B_1,B_2}) = 0. \end{aligned} \tag{4.24}$$

Before establishing (4.21) we need to record the following fact. Let $A, B \in \mathcal{B}(\mathbb{R})$ with $A \subset B$, then

$$\begin{aligned} &\langle M(A)\rangle_t(w) \leq \langle M(B)\rangle_t(w), \quad w \notin N_B \\ &\text{with } P(N_B) = 0. \end{aligned} \tag{4.25}$$

For, write B as $B = A\dot{\bigcup}(B\backslash A)$. Then $\langle M(B)\rangle_t = \langle M(A\dot{\bigcup}(B\backslash A))\rangle_t = \langle M(A)\rangle_t + \langle M(B\backslash A) >_t$, where the second equality holds outside a null set N_B by (4.24). Since the sharp bracket is zero at 0 and increasing in the index t, the inequality in (4.25) follows. We are now in a position to demonstrate (4.21). For $t_1 \leq t_2$, we have

$$\langle M(x_1, x_2]\rangle_{t_1}(w) \leq \langle M(x_1, x_2]\rangle_{t_2}(w), \quad \forall\, w \in \Omega. \tag{4.26}$$

Using the almost sure additivity of the sharp bracket in (4.24), we may rewrite (4.26) as

$$\begin{aligned} &\langle M(-\infty, x_2]\rangle_{t_1}(w) - \langle M(-\infty, x_1]\rangle_{t_1}(w) \leq \\ &\leq \langle M(-\infty, x_2]\rangle_{t_2}(w) - \langle M\rangle(-\infty, x_1]\rangle_{t_2}(w), \\ &\text{for } w \notin N_{x_1,x_2}, \text{ with } P(N_{x_1,x_2}) = 0. \end{aligned} \tag{4.27}$$

That is,

$$F_{t_1}(x_2)(w) - F_{t_1}(x_1)(w) \leq F_{t_2}(x_2)(w) - F_{t_2}(x_1)(w), \quad w \notin N_{x_1,x_2}. \tag{4.28}$$

Or, rearranging (4.28) we have

$$F_{t_2}(x_2)(w) + F_{t_1}(x_1)(w) - F_{t_1}(x_2)(w) - F_{t_2}(x_1)(w) \geq 0$$

for $w \notin N_{x_1,x_2}$. This means that

$$\Delta_D F(w) \geq 0, \quad w \notin N_{x_1,x_2} \equiv N_D.$$

This completes the proof of the lemma. ■

REMARK (4.2.3). The random function $F_t(x)(w)$ is right continuous in t (for each x and w fixed) because it is defined by a sharp bracket. However, F may not be right continuous in x. What we intend to do in the next proposition is to introduce a version of F, which, for almost all w, will be a distribution function in t and x.

We will then be able to obtain a unique measure corresponding to this distribution function by virtue of theorem (4.1.2).

PROPOSITION (4.2.4). Let $\{M_t(B), (\mathcal{F}_t)_{t\geq 0}, B \in \mathcal{B}(\mathbb{R})\}$ be an E-valued orthogonal martingale measure which satisfies (4.13). Then corresponding to the random function $F_t(x)(w)$ in lemma (4.2.2) there is a random function $H : \mathbb{R}_+ \times \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

(i) $H_t(x)(w)$ is right continuous in (t, x) for each $w \in \Omega$.

(ii) $P(H_t(x) = F_t(x),\ \forall\ t) = 1$.

(iii) There is a single null set $\widetilde{N}$ such that for any bounded rectangle $D = (t_1, t_2] \times (x_1, x_2]$ we have

$$\Delta_D H(w) \geq 0, \quad w \notin \widetilde{N}.$$

Proof. Let $\mathbb{Q}$ denote the set of rationals. By virtue of (4.25) and the fact that F is increasing and right continuous in t, we deduce that $F_t(x)(w)$ is increasing in $(t, x) \in \mathbb{R}_+ \times \mathbb{Q}$ outside of a negligible set N. Define now a function $G : \mathbb{R}_+ \times \Omega \times \mathbb{Q} \longrightarrow [0, +\infty]$ as follows:

$$G_t(x)(w) \quad = \begin{cases} F_t(x)(w), & w \notin N \\ 0, & w \in N. \end{cases} \tag{4.29}$$

Then G is increasing in $(t, x) \in \mathbb{R}_+ \times \mathbb{Q}$ everywhere. Let the function $H : \mathbb{R}_+ \times \Omega \times \mathbb{R} \longrightarrow [0, +\infty]$ be defined by

$$H_t(x)(w) \quad = \quad \lim_{\substack{t' \searrow t, r \searrow x \\ t' \in \mathbb{R}, r \in \mathbb{Q}}} G_{t'}(r)(w). \tag{4.30}$$

It is clear that H is 0 on N. We claim that H is right continuous in (t, x), for each w fixed. Let $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ and let $t_n \searrow t$, $x_n \searrow x$. Then we may obtain sequences (t'_n) and (x'_n) with $t'_n, x'_n \in \mathbb{Q}$ and $t'_n \downarrow t$, $x'_n \downarrow x$ with $t'_n \geq t_n$, $x'_n \geq x_n$. We have

$$G_{t'_n}(x'_n)(w) \geq H_{t_n}(x_n)(w) \geq H_t(x)(w), \quad \forall\ w \in \Omega. \tag{4.31}$$

The inequalities in (4.31) follow directly from definitions. Taking limits as $n \longrightarrow \infty$ in (4.31) we obtain

$$H_t(x)(w) = \lim_{n\longrightarrow\infty} G_{t'_n}(x'_n)(w) \geq \lim_{n\longrightarrow\infty} H_{t_n}(x_n)(w) \geq H_t(x)(w). \tag{4.32}$$

It follows from (4.32) that

$$\lim_{n\longrightarrow\infty} H_{t_n}(x_n)(w) = H_t(x)(w), \quad \forall\ w \in \Omega. \tag{4.33}$$

Remember, (4.32) is valid $\forall\ w \in \Omega$. This proves statement (i) of the theorem. We now prove statement (ii). Let $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. Then for any $t' > t$ and $r > x$ with $r \in \mathbb{Q}$ we have

$$F_t(x)(w) \leq F_{t'}(r)(w), \quad w \notin N_x, \tag{4.34}$$

where the negligible set, N_x, depends on x. Note that in (4.34) $F_{t'}(r)(w) = G'_t(r)(w)$ as $r \in \mathbb{Q}$ and $w \notin N_x$. Let $r \searrow x$ through the rationals and let $t' \searrow t$, then take limits in (4.34) to obtain

$$F_t(x)(w) \leq H_t(x)(w), \quad w \notin N_x. \tag{4.35}$$

Multiplying both sides of (4.35) by 1_A, $A \in \mathcal{F}$, and taking expectations we have

$$E[1_A F_t(x)] \leq E[1_A H_t(x)], \forall t \in \mathbb{R}. \tag{4.36}$$

We next prove the reverse inequality in (4.36). Let (t_n) and (x_n) be sequences in $\mathbb{Q}$ such that $t_n \searrow t$ and $x_n \searrow x$. We have

$$1_A H_t(x) \leq 1_A F_{t_n}(x_n) = 1_A[F_{t_n}(x) + F_{t_n}(x_n) - F_{t_n}(x)]. \tag{4.37}$$

The inequality in (4.37) is valid outside the negligible set N (see (4.29)). Taking expectations in (4.37) we have

$$\begin{aligned} E[1_A H_t(x)] &\leq E[1_A F_{t_n}(x)] + E[\langle M\rangle(x, x_n])_{t_n}] \\ &\leq E[1_A F_{t_n}(x)] + E[\langle M(x, x_n]\rangle_\infty] \\ &= E[1_A F_{t_n}(x)] + \mu(x_1, x_n]), \end{aligned} \tag{4.38}$$

where μ is as in (4.15). Let $n \longrightarrow \infty$. Then, $\mu((x, x_n]) \longrightarrow 0$ as μ is a σ-additive measure (by lemma (4.2.1)) and $E[1_A F_{t_n}(x)] \longrightarrow E[1_A F_t(x)]$ by monotone convergence (as $F_{t_n}(x)(w) \searrow F_t(x)(w)$, for each $w \in \Omega$). Hence, from (4.38), we obtain

$$E[1_A H_t(x)] \leq E[1_A F_t(x)] \quad \forall t \in \mathbb{R}. \tag{4.39}$$

Combining (4.36) and (4.39) we have

$$E[1_A(H_t(x) - F_t(x))] = 0, \quad \forall\, t \in \mathbb{R},\ \forall\, A \in \mathcal{F}. \tag{4.40}$$

It follows that for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ we have

$$H_t(x) = F_t(x) \quad \text{w.a.s} \tag{4.41}$$

Infact by the right continuity of F (and of H) with respect to the index t there is a single null set, say V, such that

$$H_t(x)(w) = F_t(x)(w) \quad \forall\, t \in \mathbb{R}, w \notin V. \tag{4.42}$$

Rewriting (4.42) we have

$$P(H_t(x) = F_t(x),\ \forall\, t \in \mathbb{R}) = 1. \tag{4.43}$$

This proves (ii) of the theorem. It remains to prove the statement in (iii). We can infact do better than (4.42). Since the rationals are countable, we can find a single null set, say V', such that

$$H_t(x)(w) = F_t(x)(w), \quad \forall\, t \in \mathbb{R}, \quad \forall\, x \in \mathbb{Q}, \ w \notin V'. \tag{4.44}$$

Denote by $\overline{N}$ the negligible set $\overline{N} = \bigcup_{x_1, x_2 \in \mathbb{Q},\ x_1 \leq x_2} N_{x_1, x_2}$, where the negligible sets N_{x_1, x_2} are given in (4.27). Then if D is any bounded rectangle of the form $D = (t_1, t_2] \times (x_1, x_2]$ with $x_1, x_2 \in \mathbb{Q}$ and $t_1, t_2 \in \mathbb{R}$ we have by (4.44) that

$$\Delta_D H(w) = \Delta_D F(w), \quad w \notin V'. \tag{4.45}$$

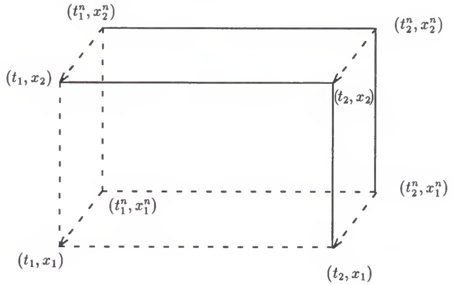
From (4.21) (of lemma (4.2.2)) we have

$$\Delta_D F(w) \geq 0, \quad w \notin \overline{N} \tag{4.46}$$

Denote by $\widetilde{N}$ the negligible set $\widetilde{N} = \overline{N} \cup V'$. Combining (4.45) and (4.46) (and keeping (4.44) in mind) we deduce that

$$\begin{aligned} &\Delta_D H(w) \geq 0, \quad \forall\, D = (t_1, t_2] \times (x_1, x_2] \ \text{ with } \ x_1, x_2 \in Q \\ &\text{and } w \notin \widetilde{N}. \end{aligned} \tag{4.47}$$

Now let D be an arbitrary bounded rectangle, $D = (t_1, t_2] \times (x_1, x_2]$, $t_1, t_2 \in \mathbb{R}_+$ and $x_1, x_2 \in \mathbb{R}$. We may approximate D from above by a sequence of rectangles D_n of the form $D_n = (t_1^n, t_2^n] \times (x_1^n, x_2^n]$, where $t_1^n, t_2^n, x_1^n, x_2^n \in Q$ and $t_1^n \searrow t_1$, $t_2^n \searrow t_2$, $x_1^n \searrow x_1$ and $x_2^n \searrow x_2$ (see the picture below).



For each n, we have $\Delta_{D_n} H(w) \geq 0$, $w \notin \widetilde{N}$ (by (4.47)). Then use the right continuity of H (statement (i)) to conclude that

$$\Delta_D H(w) \geq 0 \quad w \notin \widetilde{N}. \tag{4.48}$$

Observe that in (4.48) the same negligible set, $\widetilde{N}$, works for all D. This completes the proof of the proposition. ∎

Now that we have a distribution H which satisfies (iii) in proposition (4.2.4), we may obtain (via theorem (4.1.2) a family of measures $\{\eta_w\}_{w\in\Omega}$, each η_w corresponding to $H.(\cdot)(w)$. This together with lemma (4.1.5) will provide us with the existence of a finite positive measure on $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$, which we will use in the proof our main theorem (theorem (4.2.8)) to deduce that I_M can be extended to a σ-additive measure on $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$.

LEMMA (4.2.5). Let the sets D and $\widetilde{N}$ be as in proposition (4.2.4). Then corresponding to the random function H in proposition (4.2.4) we have the following:

(a) There exists a family $(\eta_w)_{w\in\Omega}$ of positive measures on $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) \equiv \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$ such that for each $w \in (\widetilde{N})^c$ the equality below holds

$$\eta_w(D) = \Delta_D H(w). \tag{4.49}$$

and

(b) There is a positive and finite σ-additive measure $\beta : \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F} \otimes B(\mathbb{R}) \longrightarrow \mathbb{R}_+$ given by

$$\beta(C) = \int_\Omega \left[\int_{\mathbb{R}_+ \times \mathbb{R}} 1_C(t, w, x) \eta_w(dt, dx) \right] P(dw) \tag{4.50}$$

for $C \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R})$.

In particular, on a cube in $\mathcal{R}$ (generating $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$) of the form $(t_1, t_2] \times A \times (x_1, x_2]$ with $A \in \mathcal{F}_{t_1}$, we may write β as

$$\begin{aligned} \beta((t_1, t_2] \times A \times (x_1, x_2]) &= \int_A \eta_w(D) P(dw) \\ &= E[1_A \eta(D)] \end{aligned} \tag{4.51}$$

where $D = (t_1, t_2] \times (x_1, x_2]$.

Proof. (a). For $w \in (\widetilde{N})^c$ we apply theorem (4.1.2) as H satisfies (i) and (iii) in proposition (4.2.4). This gives (4.49). For $w \in \widetilde{N}$ simply put $\eta_w \equiv 0$, the trivial measure. This proves (a).

(b). Observe that for each bounded rectangle D of the form $D = (t_1, t_2] \times (x_1 \times x_2]$, where $t_1, t_2 \in \mathbb{R}_+$ and $x_1, x_2 \in \mathbb{R}$, the function $\eta(D) : \Omega \longrightarrow \mathbb{R}_+$ is $\mathcal{F}$-measurable. This follows at once from the equality in (4.49) and the fact that H is predictable, hence $H_t(x)(\cdot)$ is $\mathcal{F}$-measurable for each $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$ fixed. It follows that for any set C in $B(\mathbb{R}_+ \times \mathbb{R}) \equiv \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$ the function $\eta(C)$ is $\mathcal{F}$-measurable. Hence we may apply lemma (4.1.3) (note that $\eta_w(\cdot)$ and P correspond to $\mu_2(\cdot, w)$ and μ_1 in that lemma) and lemma (4.1.5) to obtain a σ-additive measure $\beta : \mathcal{F} \otimes (\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})) \longrightarrow \mathbb{R}_+$, which is given by (4.50) ((4.50) is obtained from (4.12)). The statement in (4.51) then follows directly. At this juncture let us indicate that, $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}) \subset \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}) \equiv \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R})$ and hence it is still true that the restriction of β to $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$, that is $\beta|_{\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})}$, is a σ-additive measure.

Finally, we need to show that β is finite. That is, show that $\beta(\mathbb{R}_+ \times \Omega \times \mathbb{R})$ is finite. First consider a set of the form $(0, N] \times \Omega \times (-N, N]$. We shall show that $\beta((0, N] \times \Omega \times (-N, N])$ is bounded by the same constant for all N. Then letting $N \longrightarrow \infty$ and using the σ-additivity of β we will be able to deduce that β is finite. From (4.50) we have

$$\begin{aligned}
&\beta((0, N] \times \Omega \times (-N, N]) \\
&= \int_\Omega \left[\int_{(0,N]\times(-N,N]} 1_{(0,N]\times\Omega\times(-N,N]}(t, w, x) \eta_w(dt, dx) \right] P(dw) \\
&= \int_\Omega \eta_w((0, N] \times (-N, N]) P(dw) \\
&= E[\eta_w((0, N] \times (-N, N]].
\end{aligned} \tag{4.52}$$

From (4.49) we have

$$\eta_w((0, N] \times (-N, N]) = \Delta_{(0,N]\times(-N,N]} H(w) \quad \forall\, w \in \Omega. \tag{4.53}$$

Note that both sides of (4.53) will be zero if $w \in \widetilde{N}$. However, from (ii) of proposition (4.2.4) we know that

$$\Delta_{(0,N]\times(-N,N]} H = \Delta_{(0,N]\times(-N.N]} F \quad w.\text{a.s.} \tag{4.54}$$

Using the definition of F (see (4.20)) we have

$$\Delta_{(0,N]\times(-N,N]}F(w) = \langle M(-N,N]\rangle_N(w) - \langle M(-N,N]\rangle_0(w). \tag{4.55}$$

Since M is zero at zero (see the beginning of this section), it follows that the sharp bracket is also zero at zero (for, we have $E[|M_t(B)|_E^2 - \langle M(B)\rangle_t] = 0$, $\forall\ B \in \mathcal{B}(\mathbb{R})$). Hence (4.55) reduces to

$$\Delta_{(0,N]\times(-N,N]}F(w) = \langle M(-N,N]\rangle_N(w). \tag{4.56}$$

Taking expectations in (4.53) and using (4.54) and (4.56) we have

$$\begin{aligned} E[\eta_w((0,-N]\times(-N,N]) &= E[\langle M(-N,N]\rangle_N] \\ &\le E[\langle M(-N,N]\rangle_\infty] \\ &= \mu((-N,N]) \\ &\le \mu(\mathbb{R}) \le c \end{aligned} \tag{4.57}$$

In (4.57) μ is the finite measure from lemma (4.2.1) and c is a (finite) constant. Returning to (4.52) it follows that

$$\beta((0,N]\times\Omega\times(-N,N]) \le c, \quad \forall\ N. \qquad \blacksquare$$

It follows that β is a finite measure. This completes the proof of the lemma.

Remark (4.2.6). The finiteness of the measure β will play an important role in allowing us to extend I_M from $\mathcal{R}$ to $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ σ-additively (see proposition (4.2.7) and theorem (4.2.8)).

The next proposition deals with the extension of a Banach valued measure from a ring to the σ-algebra that is generated by the ring. We state it in the particular form that we require for the proof of the theorem (4.2.8). Although this proposition is similar in spirit to theorem 1 of Dinculeanu [12] (page 62) it does differ from this theorem by way of an important hypothesis. Thus a partially new proof is required (see step (f) in the proof of the proposition).

PROPOSITION (4.2.7). Let (X, Σ) be a measure space, Y a Banach space, $\mathcal{R}$ a ring generating Σ and $\alpha : \mathcal{R} \longrightarrow Y$ an additive set function. Let $\beta : \Sigma \longrightarrow [0, \infty)$ be a finite (positive) σ-additive measure for which we have

$$\alpha << \beta \text{ on } \mathcal{R}\ (\varepsilon - \delta \text{ form}). \tag{4.58}$$

Then α can be extended to a σ-additive measure, α_1, on Σ and we have $\alpha_1 << \beta$ on Σ (ε-δ form).

Proof. In order to carry out this extension we will need to appeal to a familiar method of converting a measure space into a metric space (see Dinculeanu [12], pages 60-63, or Rao [43, 44]). We provide the extension in several steps.

(a). Define a finite semi-distance ϱ on Σ by the equality $\varrho(A, B) = \beta(A \Delta B)$ for $A, B \in \Sigma$. In fact, we can show that (Σ, ϱ) is a complete semimetric space. Then, if we consider the quotient space $\tilde{\Sigma} = \Sigma/N$, where N is the ring of β-null sets, the function $\tilde{\varrho} : \tilde{\Sigma} \times \tilde{\Sigma} \longrightarrow [0, \infty)$ defined by

$$\tilde{e}([A], [B]) = \varrho(A, B); \quad [A], [B] \in \tilde{\Sigma}$$

is a metric and $(\tilde{\Sigma}, \tilde{\varrho})$ is a complete metric space.

(b). We claim that the additive set function $\alpha : \mathcal{R} \longrightarrow Y$ is uniformly continuous on $\mathcal{R}$ (with respect to ϱ). For let $\varepsilon > 0$ be given. Then by (4.58) we know that there exists a $\delta > 0$ such that for $D \in \mathcal{R}$ we have

$$\text{If } \beta(D) < \delta \text{ then } |\alpha(D)|_Y < \varepsilon/2. \tag{4.59}$$

Now let $A, B \in \mathcal{R}$ such that $e(A, B) < \delta$. We have

$$\begin{aligned} |\alpha(A) - \alpha(B)|_Y &= |\alpha(A - B) - \alpha(B - A)|_Y \\ &\leq |\alpha(A - B)|_Y + |\alpha(B - A)|_Y \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned} \tag{4.60}$$

Let us comment on the last inequality in (4.60). Since $A - B \subset A \Delta B$, we have $\beta(A - B) \leq \beta(A \Delta B) = \varrho(A, B) < \delta$ and hence $\alpha(A - B) < \varepsilon/2$. Likewise we have

$\alpha(B - A) < \varepsilon/2$. This proves that α is uniformly continuous (with respect to ϱ) on R.

(c). The ring $\mathcal{R}$ (which generates the σ-algebra Σ) is dense in Σ for the metric ϱ. This statement is nothing really new for it is just a restatement of a standard fact in measure theory (see, for example, proposition 2.16 (iii), page 50, Rao [44]) which we give below:

An element A is in Σ iff given $\varepsilon > 0$ there exists an element $A^\varepsilon \in \mathcal{R}$ such that $\beta(A \Delta A^\varepsilon) < \varepsilon$.

That is $\varrho(A, A^\varepsilon) < \varepsilon$. This proves step (c).

(d). Since α is uniformly continuous on $\mathcal{R}$ (step (b)), which is dense in Σ (step (c)), it, that is α, can be extended to a uniformly continuous function $\alpha_1 : \Sigma \longrightarrow Y$.

(e). The extension α_1 is additive. The proof of this step is in Dinculeanu [12] (page 62). We reproduce it here as it is brief. It is shown there that the operations $C \cup D$ and $C - D$ are uniformly continuous (for ϱ). Thus let A and B be two disjoint sets in Σ. Let (A_n) and (B_n) be two sequences of sets in $\mathcal{R}$ such that $\varrho(A_n, A) \longrightarrow 0$ and $\varrho(B_n, B) \longrightarrow 0$. Then $\varrho(A_n \cup B_n, A \cup B) \longrightarrow 0$ and $\varrho(A_n - B_n, A - B) \longrightarrow 0$. However, as $A \cap B = \emptyset$, the last convergence is $\varrho(A_n - B_n, A) \longrightarrow 0$. Since α is additive on $\mathcal{R}$, we have

$$\alpha(A_n \cup B_n) = \alpha((A_n - B_n) \cup B_n) = \alpha(A_n - B_n) + \alpha(B_n) \tag{4.61}$$

as $(A_n - B_n)$ and B_n are disjoint. Taking limits (4.61) becomes

$$\alpha_1(A \cup B) = \alpha_1(A) + \alpha_1(B).$$

This proves step (e).

(f). The finitely additive set function α_1 is absolutely continuous with respect to β on Σ ($\varepsilon - \delta$ form). For the proof of this step let $\varepsilon > 0$ be given and use the fact that $\alpha << \beta$ on $\mathcal{R}$ (see (4.58)) to find a $\theta > 0$ such that

$$\beta(A) < \theta \Longrightarrow |\alpha(A)|_Y < \varepsilon/2 \text{ for } A \in \mathcal{R}. \tag{4.62}$$

We now show that $\alpha_1 << \beta$ on Σ. Let $C \in \Sigma$ be such that $\beta(C) \leq \delta < \theta$. We claim that this number δ suffices for the $\varepsilon - \delta$ absolute continuity of α_1 with respect to β. From part (c) there exists a sequence (C_n) in $\mathcal{R}$ such that $\varrho(C_n, C) \longrightarrow 0$ as $n \longrightarrow \infty$. Now let $\eta > 0$ be any number such that $\eta < \theta - \delta$. Then there exists a number N_1 such that $\forall\, n \geq N_1$ we have

$$\beta(C_n \Delta C) = \varrho(C_n, C) \leq \eta. \tag{4.63}$$

Since $\varrho(C_n, C) \longrightarrow 0$ the continuity of α_1 implies that $|\alpha_1(C_n) - \alpha_1(C)|_Y \longrightarrow 0$ as $n \longrightarrow \infty$. Thus, choose a number N_2 such that

$$|\alpha_1(C_n) - \alpha_1(C)|_Y \leq \varepsilon/2, \quad \forall\, n \geq N_2. \tag{4.64}$$

Let us quickly observe that, for $n \geq N_1$, we have

$$\begin{aligned} \beta(C_n) &= \beta((C_n - C)\dot{\bigcup}(C_n \cap C)) \\ &\leq \beta(C_n - C) + \beta(C_n \cap C) \\ &\leq \beta(C_n \Delta C) + \beta(C) \leq \eta + \delta < \theta. \end{aligned}$$

Hence by (4.62) we obtain

$$|\alpha_1(C_n)|_Y = |\alpha(C_n)|_Y < \varepsilon/2, \quad \forall\, n \geq N_1. \tag{4.65}$$

Finally, we have

$$\begin{aligned} |\alpha_1(C)|_Y &\leq |\alpha_1(C) - \alpha_1(C_n)|_Y + |\alpha_1(C_n)|_Y \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \forall\, n \geq \max(N_1, N_2). \end{aligned} \tag{4.66}$$

We used (4.64) and (4.65) for (4.66). This proves that $\alpha_1 << \beta$ on Σ ($\varepsilon - \delta$ form).

(9) It follows from step (f) that α_1 is countably additive on Σ.

This completes the proof of the proposition. ∎

We are now in a position to state and prove the main theorem of this section. It is our much awaited result concerning the 2-summability of orthogonal martingale measures.

THEOREM (4.2.8). Let $\{M_t(B); (\mathcal{F}_t)_{t\geq 0},\ B \in \mathcal{B}(\mathbb{R})\}$ be an E-valued orthogonal martingale measure which satisfies (4.13). Then we have the following.

(i) There is a positive (and finite) σ-additive measure $\beta : \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}) \longrightarrow [0, \infty)$, which on a cube in $\mathcal{R}$ of the form $(s,t] \times A \times (x;y]$ with $A \in \mathcal{F}_s$ is given by

$$\beta((s,t] \times A \times (x,y]) \equiv E[1_A(\langle M(x,y]\rangle_t - \langle M(x,y]\rangle_s)] \tag{4.67}$$

(ii) The finitely additive stochastic measure $I_M : \mathcal{R} \longrightarrow L_E^2$ is absolutely continuous with respect to β on $\mathcal{R}$ ($\varepsilon - \delta$ form). In fact we have

$$||I_M(C)||_{L_E^2} = \sqrt{\beta(C)} \;\text{ for }\; C \in \mathcal{R} \tag{4.68}$$

(iii) I_M can be extended to a σ-additive measure on $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ with bounded semivariation relative to $(\mathbb{R}, L_E^2)$ given by

$$(\tilde{I}_M)_{\mathbb{R},L_E^2}(D) = ||I_M(D)||_{L_E^2} \;\text{ for }\; D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}) \tag{4.69}$$

Proof. The existence of the measure β was demonstrated in lemma (4.2.5) (which was proved under the hypothesis of proposition (4.2.4)). Thus in order to establish (i) it remains to check (4.67). By virtue of (4.51) and (4.49) in lemma (4.2.5) we have

$$\begin{aligned}
&\beta((s,t] \times A \times (x,y]) = E[1_A\eta(D)] = E[1_A\Delta_D H] \\
&= E[1_A\Delta_D F] = E[1_A(F_t(y) + F_s(x) - F_t(x) - F_s(y))] \\
&= E[1_A(\langle M(x,y]\rangle_t - \langle M(x,y]\rangle_s]
\end{aligned} \tag{4.70}$$

The third equality in (4.70) is by part (ii) of proposition (4.2.4)- - -note that the null set in that proposition plays no role as we are taking expectations. The last step is by lemma (4.2.2)- -again, null sets do not count. This proves statement (i) of the theorem.

Next we prove statement (ii). To do this we need only establish (4.68). Recall that $\mathcal{R}$ is the finite (disjoint) union of cubes of the form: (a) $(s,t] \times A \times (x,y]$, $A \in \mathcal{F}_s$ and (b) $[0_A] \times (x,y]$, $A \in \mathcal{F}_0$. We will first show that (4.68) holds for a single cube and then that it also holds for a disjoint union of cubes in $\mathcal{R}$. At this juncture we remind the reader that I_M is a L_E^2-valued finitely additive measure (see lemma (1.3.3) and

section 2, chapter 2). Now consider a cube $[0_A]\times(x,y]$. Then $||I_M([0_A]\times(x,y]||_{L^2_E}=0$ (use no. (2.1) along with the fact that $M_0(B)=0, \forall B\in\mathcal{B}(\mathbb{R})$). We may compute $\beta([0_A]\times(x,y])$ as

$$\beta([0_A]\times(x,y]) = \lim_{n\longrightarrow\infty}\beta\left(\left(\frac{-1}{n},0\right]\times A\times(x,y]\right) = 0. \tag{4.71}$$

The equality in (4.70) and the knowledge that the sharp bracket in zero at 0 was used in (4.71). In computing (4.71) we implicitly extended the index t for M over all reals by setting $M_t(B)=M_0(B)=0$, $\forall\ t<0$, $\forall\ B\in\mathcal{B}(\mathbb{R})$ and similarly we considered the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ by setting $\mathcal{F}_t=\mathcal{F}_0$ for $t<0$. Hence (4.68) holds for the cube $[0_A]\times(x,y]$, $A\in\mathcal{F}_0$.

Now suppose the cube in $\mathcal{R}$ is of the form $(s,t]\times A\times(x,y]$, $A\in\mathcal{F}_s$. For the computation that follows we may (and do) replace $(x,y]$ by any set B in $\mathcal{B}(\mathbb{R})$. We have

$$\begin{aligned}
&E[(I_M((s,t]\times A\times B))^2] = E[1_A(M_t(B)-M_s(B))^2]\\
&= E[1_A(|M_t(B)|_E^2 - 2M_t(B)M_s(B) + |M_s(B)|_E^2)]\\
&= E[1_A(|M_t(B)|_E^2 - |M_s(B)|_E^2 - 2M_s(B)(M_t(B)-M_s(B)))]\\
&= E[1_A(|M_t(B)|_E^2 - |M_s(B)|_E^2)] - 2E[1_A(M_s(B)[M_t(B)-M_s(B)])]\\
&= E[1_A(|M_t(B)|^2 - |M_s(B)|^2)] = E[1_A(\langle M(B)\rangle_t - \langle M(B)\rangle_s)]
\end{aligned} \tag{4.72}$$

The product (of $M_t(B)$ and $M_s(B)$, etc.) means, of course, the inner product in E. The second term on the right hand side of the fourth equality is zero because

$$E[1_A M_s(B)[M_t(B)-M_s(B)]] = E[1_A M_s(B)E[M_t(B)-M_s(B)|\mathcal{F}_s]] = 0$$

as $M_\bullet(B)$ is a martingale. The sixth equality in (4.72) is true since $|M_.(B)|_E^2$ is a submartingale. To be more explicit, by the Doob-Meyer decomposition we have

$$|M_t(B)|_E^2 = Y_t(B) + \langle M_t(B)\rangle, \tag{4.73}$$

where $Y(B)$ is a martingale (of class (D)). A simple computation will yield the equality below:

$$I_{|M|_E^2}((s,t]\times A\times B) = I_Y((s,t]\times A\times B) + I_{\langle M\rangle}((s,t]\times A\times B) \tag{4.74}$$

Taking expectations in (4.74) will yield the sixth equality in (4.72) since

$$\begin{aligned} E(I_Y(s,t] \times A \times B)) &= E[1_A(Y_t(B) - Y_s(B)] \\ &= E[1_A E[Y_t(B) - Y_s(B)|\mathcal{F}_s]] = 0 \end{aligned}$$

as $Y_\cdot(B)$ is a martingale. Thus (4.68) follows from (4.72) (and (4.67)) for the cube $(s,t] \times A \times B$.

Finally, we prove the equality in (4.68) for an arbitrary element C in $\mathcal{R}$. Thus let $C = \bigcup_{i=1}^n C_i$, where the C_i are mutually disjoint for $i = 1, \dots, n$ and each C_i is of the form $C_i = (t_1^i, t_2^i] \times A_i \times (s_1^i, s_2^i]$ with $A_i \in \mathcal{F}_{t_1^i}$. Since β is a σ-additive measure we have $\beta(C) = \sum_{i=1}^n \beta(C_i)$. We have indicated at the start of this chapter that I_M is finitely additive on $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$. Thus $I_M(C) = \sum_{i=1}^n I_M(C_i)$ in L_E^2. That is,

$$||I_M(C)||_{L_E^2} = \left\| \sum_{i=1}^n I_M(C_i) \right\|_{L_E^2}.$$

In order to check (4.68) it suffices to show that

$$||I_M(C)||_{L_E^2}^2 = \sum_{i=1}^n ||I_M(C_i)||_{L_E^2}^2. \tag{4.75}$$

Writing out the left hand side of (4.75) we have:

$$\begin{aligned} ||I_M(C)||_{L_E^2}^2 &= \left\| \left(\sum_{i=1}^n I_M(C_i) \right) \right\|_{L_E^2}^2 \\ &= \sum_{i=1}^n ||(I_M(C_i))||_{L_E^2}^2 + 2\sum_{i<j} E[I_M(C_i)I_M(C_j)] \\ &= \beta(C) + 2\sum_{i<j} E[I_M(C_i)I_M(C_j)]. \end{aligned} \tag{4.76}$$

The first term in the last equality in (4.76) obtains from (4.72) (use (4.67)) and the fact that β is a measure. As for the terms of the form $E[I_M(C_i)I_M(C_j)]$ in (4.76), we will now demonstrate that they can be written as

$$E[I_M(C_i)I_M(C_j)] = \beta(C_i \cap C_j), \quad i \neq j. \tag{4.77}$$

However, as $C_i \cap C_j = \emptyset$, for $i \neq j$, the terms in (4.77) will be zero (β is a measure). Then (4.75) is immediate.

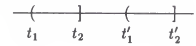
We shall now begin our proof of (4.77). Consider two disjoint cubes C_1 and C_2 with $C_1 = (t_1 t_2] \times A \times B$, $C_2 = (t'_1, t'_2] \times A' \times B'$, where $A \in \mathcal{F}_{t_1}$, $A' \in \mathcal{F}_{t'_1}$, $B = (s_1, s_2]$ and $B' = (s'_1, s'_2]$. Let $D = B \cap B'$. We may write C_1 and C_2 as: $C_1 = C_{1,1} \dot{\bigcup} C_{1,2}$ and $C_2 = C_{2,1} \dot{\bigcup} C_{22}$, where $C_{1,1} = (t_1, t_2] \times A \times B \backslash B'$, $C_{1,2} = (t_1, t_2] \times A \times D$ and $C_{2,1} = (t'_1, t'_2] \times A' \times B' \backslash B$, $C_{2,2} = (t'_1, t'_2] \times A' \times D$. Then $I_M(C_i) = I_M(C_{i1}) + I_M(c_{i2})$, $i = 1, 2$, where this equality is in L^2_E (and hence also in L^1_E). Then,

$$I_M(C_1) I_M(C_2) = I_M(C_{1,1}) I_M(C_{2,1}) + I_M(C_{1,1}) I_M(C_{2,2})$$
$$+ I_M(C_{1,2}) I_M(C_{2,1}) + I_M(C_{1,2}) I_M(C_{2,2}).$$

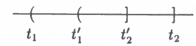
Taking expectations above and employing the orthogonality of M one easily checks that the expectations of the first three terms on the right hand side are zero. Thus $E(I_M(C_1) I_M(C_2))$ is equal to

$$E[I_M(C_{1,2}) I_M(C_{2,2})] = E[I_M((t_1, t_2] \times A \times D) \times$$
$$I_M((t'_1, t'_2] \times A' \times D)] = E[1_{A \cap A'} (M_{t_2}(D) - M_{t_1}(D)) \times \tag{4.78}$$
$$(M_{t'_2}(D) - M_{t'_1}(D))], \quad \text{where} \quad A \cap A' \in \mathcal{F}_{t_1 \vee t'_1}.$$

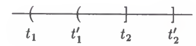
In order to confirm (4.77) we check three cases corresponding to the relative positions of the time intervals $(t_1, t_2]$ and $(t'_1, t'_2]$ with respect to each other. The three cases are (in pictures)

(a) 

; $A \cap A' \in \mathcal{F}_{t_1 \vee t'_1} = \mathcal{F}_{t'_1}$

(b) 

; $A \cap A' \in \mathcal{F}_{t'_1}$

(c) 

; $A \cap A' \in \mathcal{F}_{t'_1}$

We now compute $E[I_M(C_1) I_M(C_2)]$ for each of these cases.

Case (a). We have

$$\begin{aligned}
&E[I_M(C_1)I_M(C_2)] = E[I_M(C_{1,2})I_M(C_{2,2})] \\
&= E[1_{A\cap A'}M_{t_2}(D)[M_{t'_2}(D) - M_{t'_1}(D)]] - \\
&E[1_{A\cap A'}M_{t_1}(D)[M_{t'_2}(D) - M_{t'_1}(D)]] \\
&= E[1_{A\cap A'}M_{t_2}(D)E[M_{t'_2}(D) - M_{t'_1}(D)|\mathcal{F}_{t'_1}]] \\
&- E[1_{A\cap A'}M_{t_1}(D)E[M_{t'_2}(D) - M_{t'_1}(D)|\mathcal{F}_{t'_1}]] \\
&= 0.
\end{aligned} \tag{4.79}$$

But $\beta(C_{1,2} \cap C_{2,2}) = \beta(\emptyset) = 0$. Thus case (a) verifies (4.77).

Case (b). Here

$$\begin{aligned}
I_M(C_{1,2})I_M(C_{2,2}) &= 1_{A\cap A'}[(M_{t_2}(D)) - M_{t_1}(D)) \times (M_{t'_2}(D) - M_{t'_1}(D))] \\
&= 1_{A\cap A'}\{[(M_{t_2}(D) - M_{t'_2}(D)) \times (M_{t'_2}(D) - M_{t'_1}(D))] \\
&\quad +[(M_{t'_2}(D) - M_{t'_1}(D)) \times (M_{t'_2}(D) - M_{t'_1}(D))] \\
&\quad +[(M_{t'_1}(D) - M_{t_1}(D)) \times (M_{t'_2}(D) - M_{t'_1}(D))]\}.
\end{aligned} \tag{4.80}$$

Take expectations in (4.80). The expectations of the first and third terms in the brackets is zero, by a similar reasoning to case (a). For the expectation of the second term in brackets we write it as follows.

$$E[1_{A\cap A'}M_{t'_2}(D)(M_{t'_2}(D) - M_{t'_1}(D))] - E[1_{A\cap A'}M_{t'_1}(D)(M_{t'_2}(D) - M_{t'_1}(D))].$$

The second term is zero because it can be written as

$$E[1_{A\cap A'}M_{t'_1}(D)E[M_{t'_2}(D) - M_{t'_1}(D)|\mathcal{F}_{t'_1}]] = 0$$

(remember: $M(D)$ is a martingale and $t_2 \geq t'_2 \geq t'_1$ for case (b)). The first term can be written as

$$\begin{aligned}
&E[1_{A\cap A'}M_{t'_2}(D)M_{t'_2}(D)] - E[1_{A\cap A'}M_{t'_2}(D)M_{t'_1}(D)] \\
&= E[1_{A\cap A'}M_{t'_2}(D)E[M_{t_2}(D)|\mathcal{F}_{t'_2}]] - E[1_{A\cap A'}M_{t'_1}(D)E[M_{t'_2}(D)|\mathcal{F}_{t'_1}]] \\
&= E[1_{A\cap A'}M_{t'_2}(D)M_{t'_2}(D)] - E[1_{A\cap A'}M_{t'_2}(D)M_{t'_1}(D)] \\
&= E[1_{A\cap A'}[|M_{t'_2}(D)|_E^2 - |M_{t'_1}(D)|_E^2]] \\
&= E[1_{A\cap A'}(\langle M(D)\rangle_{t'_2} - \langle M(D)\rangle_{t'_1})] \\
&= \beta((t'_1, t'_2] \times A \cap A' \times D) = \beta(C_1 \cap C_2).
\end{aligned}$$

The fourth equality above is due to (4.72) and the fifth equality follows from (4.67). Thus,

$$E[I_M(C_1)I_M(C_2)] = \beta(C_1 \cap C_2)$$

and this gives (4.77).

Case (c). Write $I_M(C_{1,2}) \cdot I_M(C_{2,2})$ as

$$\begin{aligned} &1_{(A\cap A'}\{[M_{t_2}(D) - M_{t_1'}(D)] \times [M_{t_2'}(D) - M_{t_2}(D)] + \cdots \\ &+[M_{t_2}(D) - M_{t_1'}(D)] \times [M_{t_2}(D) - M_{t_1'}(D)] + \cdots \\ &+[M_{t_1'}(D) - M_{t_1}(D)] \times [M_{t_2'}(D) - M_{t_2}(D)] + \cdots \\ &+[M_{t_1'}(D) - M_{t_1}(D)] \times [M_{t_2}(D) - M_{t_1'}(D)]\}. \end{aligned} \tag{4.81}$$

Take expectations in (4.81). Following the ideas in case (b) the expectation of the second term in the bracket is equal to

$$\beta((t_1', t_2] \times A \cap A' \times D) = \beta(C_1 \cap C_2).$$

Each of the other terms in the bracket has expectation zero. Let us detail these facts. We write the expectation of the first term in (4.81) as

$$\begin{aligned} E[1_{A\cap A'}M_{t_2}(D)E[M_{t_2'}(D) - M_{t_2}(D)|\mathcal{F}_{t_2}]] - E[1_{A\cap A'}M_{t_1'}(D) \\ E[M_{t_2'}(D) - M_{t_2}(D)|\mathcal{F}_{t_2}]] = 0. \end{aligned}$$

As for the expression of the third term in (4.81), write it as

$$\begin{aligned} E[1_{A\cap A'}M_{t_1'}(D)E[M_{t_2'}(D) - M_{t_2}(D)|\mathcal{F}_{t_2}]] - E[1_{A\cap A'}M_{t_1}(D) \\ E[M_{t_2'}(D) - M_{t_2}(D)|\mathcal{F}_{t_2}]] = 0. \end{aligned}$$

Finally write the expectation of the fourth term in (4.81) as

$$\begin{aligned} E[1_{A\cap A'}M_{t_1'}(D)E[M_{t_2}(D) - M_{t_1'}(D)|\mathcal{F}_{t_1'}]] - E[1_{A\cap A'}M_{t_1}(D) \\ E[M_{t_2}(D) - M_{t_1'}(D)|\mathcal{F}_{t_1'}]] = 0. \end{aligned}$$

The above computations imply that the expectation of (4.81), which is $E[I_M(C_1)I_M(C_2)]$ yields (4.77). This completes case (c). The orthogonality condition in (4.75) now holds. Hence we have

$$||I_M(C)||^2_{L^2_E} = \beta(C), \quad C \in \mathcal{R}.$$

Statement (ii) is now proved.

We now come to the proof of statement (iii) of the theorem. Since $I_M << \beta$ on $\mathcal{R}$ ($\varepsilon - \delta$ form), by virtue of proposition (4.2.7) we may extend I_M to a L^2_E-valued σ-additive measure on $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$. To show that M is 2-summable relative to $(\mathbb{R}, L^2_E)$ it suffices, by proposition (2.3.17), to prove that I_M has bounded semivariation, $(\tilde{I}_M)_{\mathbb{R},L^2_E}$, on $\mathcal{R}$. In this direction let $C \in \mathcal{R}$. Let $(C_i)_{i=1}^n$ be a partition of C, where each C_i has the form $C_i = (t_i^2, t_i^2] \times A_i \times (x_i^1, x_i^2]$, with $A_i \in \mathcal{F}_{t_i'}$ and let $(x_i)_{i=1}^n$ be any family of elements in $\mathbb{R}$ with $|x_i| \leq 1$, $i = 1, \ldots, n$. Then we have

$$\begin{aligned} \left\|\sum_{i=1}^n I_M(C_i)x_i\right\|^2_{L^2_E} &= \textstyle\sum_{i=1}^n ||I_M(C_i)x_i||^2_{L^2_E} \\ &\leq \sum_{i=1}^n ||I_M(C_i)||^2_{L^2_E} \\ &= \left\|\sum_{i=1}^n I_M(C_i)\right\|^2_{L^2_E} \\ &= I_M(C)||^2_{L^2_E} \end{aligned} \tag{4.82}$$

The first two equalities in (4.82) follow from (4.75) and the last equality obtains as I_M is a (σ-additive) L^2_E-valued measure. From (4.82) we conclude that

$$(\tilde{I}_M)_{\mathbb{R},L^2_E}(C) \leq ||I_M(C)||_{L^2_E}. \tag{4.83}$$

Furthermore, we have

$$||I_M(C)||^2_{L^2_E} = \beta(C) \leq \beta(\mathbb{R} \times \Omega \times \mathbb{R}_+) < \infty. \tag{4.84}$$

The finiteness in (4.84) is by lemma (4.2.5), part (b). Hence I_M has bounded semivariation, $(\tilde{I}_M)_{\mathbb{R},L^2_E}$, on $\mathcal{R}$. This completes the proof of the statement in (iii) that M is a 2-summable relative to $(\mathbb{R}, L^2_E)$. The equality in (4.69) will be proved in proposition (4.2.10). ∎

The lemma that follows will be used in proposition (4.2.10), where the equality in (4.69) is proved.

LEMMA (4.2.9).

(a) Let M and N be E-valued orthogonal martingale measures which satisfy condition (4.13). Then for any pair of disjoint sets C and D in $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ we have:

$$I_M(C) \perp I_N(D) \text{ in } L^2_E. \tag{4.85}$$

(b) If M and N in (a) are real valued and C, D are as above then for any x, y in E we have

$$I_M(C)x \perp I_N(D)y. \tag{4.86}$$

Proof. In the proof of theorem (4.2.8) (see number (4.77)) it was established that

$$I_M(C) \perp I_M(D), \quad \text{for } C, D \in \mathcal{R}, \text{ disjoint.} \tag{4.87}$$

We can use exactly the same argument to prove (4.85) in the case when the sets C and D are in $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$.

To prove (4.85) in the general case we use the fact that I_M and I_N are σ-additive measures (theorem (4.2.8)) to prove that the sets below

$$\mathcal{M}_C = \{B \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}) : I_M(C) \perp I_N(B - C)\}$$

and

$$\mathcal{M}_D = \{A \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}) : I_M(A) \perp I_N(D - A)\},$$

where C and D are arbitrary but fixed sets in $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$, are monotone classes. This gives (4.85) and part (a) is proved.

(b). Part (b) is proved the same way as (a) once we realize that Mx and Ny are E-valued orthogonal martingale measures. It is evident that Mx and Ny are martingale measures. We now check the orthogonality of Mx (the argument is the same for Ny). We will prove the orthogonality by demonstrating the following:

$$\begin{aligned}&\langle Mx, Mx\rangle(A) = |x|^2_E \langle M, M\rangle(A), \quad \text{for } A \in \mathcal{B}(\mathbb{R}) \\ &\text{and } x \in E.\end{aligned} \tag{4.88}$$

The sharp bracket $\langle Mx, Mx\rangle(A)$ in (4.88) is taken to mean $\langle M(A)x, M(A)x\rangle$; similarly $\langle M, M\rangle(A)$ means $\langle M(A), M(A)\rangle$. Since $|Mx|_E^2$ is a submartingale, the Doob-Meyer decomposition yields the equality

$$|Mx|_E^2(A) \equiv |M(A)x|_E^2 = X(A) + \langle Mx, Mx\rangle(A), \qquad A \in \mathcal{B}(\mathbb{R}). \tag{4.89}$$

The process $X(A)$ in (4.89) is a martingale and $\langle Mx, Mx\rangle(A)$ is the unique predictable increasing process associated with $|M(A)x|_E^2$. On the other hand we may write $|Mx|_E^2$ as $|M|^2|x|_E^2$ and now writing down the Doob-Meyer decomposition for $|M|^2(A) \equiv |M(A)|^2$ we have

$$|M|^2(A)|x|_E^2 \equiv |M(A)|^2|x|_E^2 = (Y(A) + \langle M, M\rangle(A))|x|_E^2. \tag{4.90}$$

The process $Y(A)$ in (4.90) is a martingale and $\langle M, M\rangle(A)$ is the unique predictable increasing process associated with $|M(A)|^2$. Since $Y(A)|x|_E^2$ is still a martingale it follows upon comparing (4.89) and (4.90) that (4.88) holds. This completes the proof of (b). ∎

PROPOSITION (4.2.10).

(a) If M is an E-valued orthogonal martingale measure, zero at zero, which satisfies condition (4.13), then we have

(1) $(\tilde{I}_M)_{\mathbb{R},L_E^2}(D) = ||I_M(D)||_{L_E^2}$ for $D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ and

(2) The set of measures $(I_M)_{\mathbb{R},L_E^2}$ is uniformly σ-additive.

(b) Assume the hypothesis in (a) holds, except that now M is real valued. Then considering $\mathbb{R}$ as embedded in $L(E, E)$ we have

(1′) $(\tilde{I}_M)_{E,L_E^2}(D) = ||I_M(D)||_{L_{\mathbb{R}}^2}$ for $D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$

and

(2′) The set of measures $(I_M)_{E,L_E^2}$ is uniformly σ-additive.

Proof. (a). Since M is 2-summable relative to $(\mathbb{R}, L^2_E)$ (theorem (4.2.8)) we know that I_M has bounded semivariation $(\tilde{I}_M)_{\mathbb{R},L^2_E}$. We now prove (1). Let $D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ and let $(D_i)_{i=1}^n$ be a family of disjoint sets in $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ such that $D = \dot{\bigcup}_{i=1}^n D_i$. Let $(y_i)_{i=1}^n$ be a family of elements in $\mathbb{R}$ with $|y_i| \leq 1$ for $i = 1, \ldots, n$. By virtue of (4.75) we have

$$\begin{aligned} \left\|\sum_{i=1}^n I_M(D_i)y_i\right\|^2_{L^2_E} &= \sum_{i=1}^n ||I_M(D_i)y_i||^2_{L^2_E} \\ &\leq \sum_{i=1}^n ||I_M(D_i)||^2_{L^2_E} \\ &= \left\|\sum_{i=1}^n I_M(D_i)\right\|^2_{L^2_E} \\ &= ||I_M(D)||^2_{L^2_E}. \end{aligned} \tag{4.91}$$

The last equality in (4.91) obtains as I_M is a (σ-additive) measure. It follows from (4.91) that

$$(\tilde{I}_M)_{\mathbb{R},L^2_E}(D) \leq ||I_M(D)||_{L^2_E}. \tag{4.92}$$

The reverse inequality in (4.92) follows by definition of the semivariation of I_M relative to $(\mathbb{R}, L^2_E)$. Statement (2) is now immediate. This completes the proof of (a).

(b). The proof for (b) is similar to (a). To make matters transparent we go through the computations in (4.91) when M is real valued and $(y_i)_{i=1}^n$ are in E, with $|y_i|_E \leq 1$. We have

$$\begin{aligned} \left\|\sum_{i=1}^n I_M(D_i)y_i\right\|^2_{L^2_E} &= \sum_{i=1}^n ||I_M(D_i)y_i||^2_{L^2_E} \\ &= \sum_{i=1}^n ||I_M(D_i)||^2_{L^2_{\mathbb{R}}} |y_i|^2_E \\ &\leq \sum_{i=1}^n ||I_M(D_i)||^2_{L^2_{\mathbb{R}}} \\ &= \left\|\sum_{i=1}^n I_M(D_i)\right\|^2_{L^2_{\mathbb{R}}} \\ &= ||I_M(D)||^2_{L^2_{\mathbb{R}}}. \end{aligned}$$

The rest is the same. This proves (b). ∎

The last result of this section is a by-product of theorem (4.2.8). It says that if M is an orthogonal martingale measure that satisfies the condition in (4.13) then $\langle M \rangle$

is summable relative to $(\mathbb{R}, L^1_{\mathbb{R}})$.

PROPOSITION (4.2.11). Let M be an E-valued orthogonal martingale measure, zero at zero, which satisfies condition (4.13). Then $\langle M\rangle$ is a process measure that is summable relative to $(\mathbb{R}, L^1_{\mathbb{R}})$. The semivariation of $I_{\langle M\rangle}$, $(\widetilde{I}_{\langle M\rangle})_{\mathbb{R},L^1_{\mathbb{R}}}$, can be computed as follows.

$$(\widetilde{I}_{\langle M\rangle})_{\mathbb{R},L^1_{\mathbb{R}}}(D) = ||I_{\langle M\rangle}||_{L^1_{\mathbb{R}}}(D), \quad D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}) \tag{4.93}$$

Proof. By properties of the sharp bracket condition (a) of definition (2.1.1) is satisfied. Next, since for each $t \geq 0$, $M - t(\cdot)$ is a L^2_E $\sigma-$additive measure, it follows from (4.17) that $\langle M(\cdot)\rangle_t$ is definition (2.1.1) is also satisfied and hence $\langle M\rangle$ is a process measure. We now show that $I_{\langle M\rangle} : \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}) \longrightarrow L^1_{\mathbb{R}}$ is a σ-additive measure with finite semivariation relative to $(\mathbb{R}, L^1_{\mathbb{R}})$. Recall from (4.67) that on a cube of the form $(s,t] \times A \times B$, with $A \in \mathcal{F}_s$ and $B \in \mathcal{B}(\mathbb{R})$ the positive (finite) measure $\beta : \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}) \longrightarrow [0,\infty)$ is given by

$$\begin{aligned} \beta((s,t] \times A \times B) &= E[1_A[\langle M\rangle_t(B) - \langle M\rangle_s(B)] \\ &= E[I_{\langle M\rangle}((s,t] \times A \times B)]. \end{aligned} \tag{4.94}$$

That is, $||I_{\langle M\rangle}((s,t] \times A \times B||_{L^1_{\mathbb{R}}} = \beta((s,t] \times A \times B)$. Let $C = (\dot{\bigcup}_{i=1} C_i) \in \mathcal{R}$, where the C_i are cubes of the form in (4.94). Then $\|I_{\langle M\rangle}(C)\|_{L^1_{\mathbb{R}}} = \|\sum_{i=1}^n I_{\langle M\rangle}(C_i)\|_{L^1_{\mathbb{R}}}$ (see section (2.2) and lemma (1.3.3)). Since the sharp bracket is increasing (and zero at zero), we have $\|\sum_{i=1}^n I_{\langle M\rangle}(C_i)\|_{L^1_{\mathbb{R}}} = \sum_{i=1}^n \|I_{\langle M\rangle}(C_i)\|_{L^1_{\mathbb{R}}}$. Thus we obtain

$$||I_{\langle M\rangle}(C)||_{L^1_{\mathbb{R}}} = \beta(C). \tag{4.95}$$

Now $I_{\langle M\rangle} : \mathcal{R} \longrightarrow L^1_{\mathbb{R}}$ is a finitely additive measure, which by (4.95) is absolutely continuous ($\varepsilon - \delta$ form) with respect to β on $\mathcal{R}$. Thus it follows, by proposition (4.2.7), that $I_{\langle M\rangle}$ can be extended to a $L^1_{\mathbb{R}}$ valued σ-additive measure on $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$. Since $I_{\langle M\rangle} \geq 0$ a.s. on $\mathcal{R}$, it follows by a monotone class argument that

$$I_{\langle M\rangle}(D) \geq 0 \ \ a.s. \ \ D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}). \tag{4.96}$$

The null set in (4.96) depends on D. We now compute the semivariation of $I_{\langle M\rangle}$ relative to $(\mathbb{R}, L^1_{\mathbb{R}})$. Let $D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ and let $(D_i)_{i=1}^n$ be a family of disjoint sets in $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ whose union is D and let $(y_i)_{i=1}^n$ be a family of elements in $\mathbb{R}$ with $|y_i| \leq 1$. Then we have

$$\begin{aligned}
\left\|\sum_{i=1}^n I_{\langle M\rangle}(D_i)y_i\right\|_{L^1_{\mathbb{R}}} &\leq \sum_{i=1}^n \|I_{\langle M\rangle}(D_i)\|_{L^1_{\mathbb{R}}} \\
&= \left\|\sum_{i=1}^n I_{\langle M\rangle}(D_i)\right\|_{L^1_{\mathbb{R}}} \\
&= \|I_{\langle M\rangle}(D)\|_{L^1_{\mathbb{R}}}.
\end{aligned} \tag{4.97}$$

The first equality in (4.97) follows as $I_{\langle M\rangle}(D_i) \geq 0$ a.s. and there are only finitely many sets D_i. The second equality is obtained as $I_{\langle M\rangle}$ is a $L^1_{\mathbb{R}}$-valued (σ-additive) measure. From (4.97) we have

$$(\widetilde{I}_{\langle M\rangle})_{\mathbb{R},L^1_{\mathbb{R}}}(D) \leq \|I_{\langle M\rangle}(D)\|_{L^1_{\mathbb{R}}}. \tag{4.98}$$

The reverse inequality in (4.98) follows from the definition of semivariation. We thus obtain (4.93). This completes the proof of the proposition. ∎

4.3 The Stochastic Integral $(H\cdot M)$ When M Is an Orthogonal Martingale Measure

In this section we shall confine our attention to a Hilbert valued orthogonal martingale measure M which satisfies the condition in (4.13).

Let E denote the Hilbert space in which M takes its values. Under these hypotheses we know from theorem (4.2.8) that M is 2-summable relative to $(\mathbb{R}, L^2_E)$. From theorem (3.6.1) we conclude that for $H \in L_{\mathbb{R},L^2_E}(M)$, the stochastic integral $(H \cdot X)$ is bounded in L^2_E. integrable martingale. Furthermore since E is reflexive (and hence $(L^2_E)^{**} = L^2_E$), we conclude by the same theorem that $(H \cdot X)$ is 2-summable relative to $(\mathbb{R}, L^2_E)$.

Our primary concern at this stage is to demonstrate that the familiar Ito isometry in the standard theory of the stochastic integral carries over to the situation at hand.

More precisely, we show that the mapping $H \longrightarrow H \cdot M$ from $L^1_{\mathbb{R},L^2_E}(\mathcal{B}, M)$ (the closure of the bounded processes in $L^1_{\mathbb{R},L^2_E}(M)$) into $L^2_E(\mu_{\langle M \rangle})$ is an isometry. The details of this statement may be found in theorem (4.3.5). We remark here that the Doléans function induced by the process measure $\langle M \rangle$ (see proposition (4.2.11), where we established that $\langle M \rangle$ is a summable process measure), that is $\mu_{\langle M \rangle}(D) = E[I_{\langle M \rangle}(D)]$, is infact a σ-additive measure (proposition (4.3.1)). This fact allows us to obtain the Ito isometry. We also show that, as alluded to earlier in section 6 of chapter 3, the stochastic integral $(H \cdot M)$ itself turns out to be an orthogonal martingale measure in the case when $H \in L^1_{\mathbb{R},L^2_E}(\mathcal{B}, M)$. The next proposition clears the path to establishing the Ito isometry.

PROPOSITION (4.3.1). Let M be an E-valued orthogonal martingale measure which satisfies the condition in (4.13). We have

(1) The Dolean function $\mu_{\langle M \rangle} : \mathcal{R} \longrightarrow \mathbb{R}_+$, where $\mu_{\langle M \rangle} = E[I_{\langle M \rangle}]$, can be extended to a positive σ-additive measure on $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$.

(2) The Doléan function associated with the submartingale $|M|^2_E$, $\mu_{|M|^2_E} : \mathcal{R} \longrightarrow \mathbb{R}_+$ can be extended to a positive σ-additive measure on $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ and we have

$$\mu_{\langle M \rangle} = \mu_{|M|^2_E}. \tag{4.99}$$

REMARK (4.3.2). It is understood that when we write $\langle M \rangle(A)$ or $\langle M, N \rangle(A)$ or $\langle M \rangle(A, B)$ for $A, B \in \mathcal{B}(\mathbb{R})$ we mean respectively $\langle M \rangle(A)$ or $\langle M(A), N(A) \rangle$ or $\langle M(A), M(B) \rangle$. Similarly when we write $|M|^2_E(A)$ we mean $|M(A)|^2_E$.

Proof (of theorem (4.3.1)). In proposition (4.2.11) we proved that $\langle M \rangle$ is a process measure that is summable relative to $(\mathbb{R}, L^1_{\mathbb{R}})$. We had there that (see number (4.94)) the positive σ-additive measure $\beta : \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}) \longrightarrow \mathbb{R}_+$ was equal to $\mu_{\langle M \rangle}$ on $\mathcal{R}$. That is,

$$\begin{aligned} \beta((s,t] \times A \times B) &= E[1_A(\langle M \rangle_t(B) - \langle M \rangle_s(B))] \\ &= E[I_{\langle M \rangle}(s,t] \times A \times B) \\ &= \mu_{\langle M \rangle}((s,t] \times A \times B) \end{aligned} \tag{4.100}$$

where $A \in \mathcal{F}_s$ and $B \in \mathcal{B}(\mathbb{R})$. We were then able to conclude that $I_{\langle M\rangle} : \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}) \longrightarrow L^1_{\mathbb{R}}$ was a σ-additive measure and that

$$I_{\langle M\rangle}(D) \geq 0 \quad w.a.s., \ D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}). \tag{4.101}$$

The null set in (4.101) may, in general, depend upon D. It then follows that $\mu_{\langle M\rangle} = E(I_{\langle M\rangle})$ is a positive σ-additive measure on $\mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$. This proves statement (1).
(2). Since $|M|^2_E$ is a submartingale which is cadlag, we may, by the Doob-Meyer decomposition write it as

$$|M|^2_E = Y + \langle M\rangle, \tag{4.102}$$

that is for each $B \in \mathcal{B}(\mathbb{R})$ we may write

$$|M_t(B)|^2_E = Y(B) + \langle M_t(B)\rangle. \tag{4.103}$$

In number (4.102), $Y.(B)$ is a martingale (of class (D)) and $\langle M(B)\rangle$ is a predictable, integrable increasing process zero at 0. We have already indicated that the sharp bracket $\langle M\rangle$ of M is a summable process measure. A simple computation will verify that (4.103) implies the following:

$$I_{|M|^2_E} = I_Y + I_{\langle M\rangle} \quad \text{on} \quad \mathcal{R}. \tag{4.104}$$

Taking expectations in (4.104) we obtain

$$\mu_{|M|^2_E} = \mu_Y + \mu_{\langle M\rangle} \quad \text{on} \quad \mathcal{R}. \tag{4.105}$$

However, note that μ_Y is 0 on $\mathcal{R}$ as Y is a martingale. Let us verify this quickly. Let $(s,t] \times A \times B$ with $A \in \mathcal{F}_s$, $B \in \mathcal{B}(\mathbb{R})$ be a cube in $\mathcal{R}$. Then we have

$$\begin{aligned} \mu_Y((s,t] \times A \times B) &= E[1_A Y_t(B) - Y_s(B)] \\ &= E[1_A E[Y_t(B) - Y_s(B)|\mathcal{F}_s]] = 0. \end{aligned} \tag{4.106}$$

Hence we have

$$\mu_{|M|^2_E} = \mu_{\langle M\rangle} \quad \text{or} \quad \mathcal{R}. \tag{4.107}$$

Return now to the proof of statement (1). A quick look at (4.100) yields

$$\beta = \mu_{|M|^2_E} \quad \text{on} \quad \mathcal{R}. \tag{4.108}$$

By the argument following (4.100) we now conclude that

$$\mu_{|M|^2_E} : \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}) \longrightarrow L^1_{\mathbb{R}}$$

is a σ-additive measure. In fact $\mu_{|M|^2_E}$ is a positive measure- -this will be immediate once we establish (4.99). But (4.99) follows at once from (4.107) and a monotone class argument, using now the fact that $\mu_{\langle M\rangle}$ and $\mu_{|M|^2_E}$ are σ-additive measures. This proves (2) and ends the proof of the theorem. ∎

REMARK (4.3.3). For $D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ we write

$$\begin{aligned} \mu_{\langle M\rangle}(D) &= E[I_{\langle M\rangle}(D)] \\ &= E\left[\int 1_D \, dI_{\langle M\rangle}\right] \\ &= E\left[\int 1_D \, d\langle M\rangle\right]. \end{aligned}$$

REMARK (4.3.4). From Proposition (4.2.11) in section 4.2 we have the following equality.

$$(\widetilde{I}_{\langle M\rangle})_{\mathbb{R},L^1_{\mathbb{R}}}(D) = ||I_{\langle M\rangle}(D)||_{L^1_{\mathbb{R}}}, \quad D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}). \tag{a}$$

Since $\mu_{|M|^2_E} = \mu_{\langle M\rangle}$ (by (4.99)), we may, by exactly the same argument given in the proof of proposition (4.2.11), conclude that the following equality holds

$$(\widetilde{I}_{|M|^2_E})(D) = ||I_{|M|^2_E}(D)||_{L^1_{\mathbb{R}}}, \; D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}). \tag{b}$$

From the proof of proposition (4.2.11) we have that $I_{\langle M\rangle}(D)$ and $I_{|M|^2_E}(D)$ are positive almost surely for each $D \in \mathcal{P}_\infty \otimes B(\mathbb{R})$. Thus we have

$$\mu_{\langle M\rangle}(D) = ||I_{\langle M\rangle}(D)||_{L^1_{\mathbb{R}}} \tag{c}$$

and

$$\mu_{|M|^2_E}(D) = ||I_{|M|^2_E}(D)||_{L^1_{\mathbb{R}}}.$$

From (a) and (b), and the above , we obtain the equality

$$\tilde{I}_{\langle M\rangle_{\mathbb{R},L^1_{\mathbb{R}}}}(D) = (\tilde{I}_{|M|^2_E})_{\mathbb{R},L^1_{\mathbb{R}}}(D). \tag{4.109}$$

Now let $S = \sum_{i=1}^{n} s_i I_{D_i}$, $D_i \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$, be a positive predictable simple function. Then computing $||\int S\ dI_{\langle M\rangle}||_{L^1_{\mathbb{R}}}$ and $||\int S\ dI_{|M|^2_E}||_{L^1_{\mathbb{R}}}$ we have

$$\begin{aligned}
\left|\left|\int S\ dI_{\langle M\rangle}\right|\right|_{L^1_{\mathbb{R}}} &= \left|\left|\sum_{i=1}^{n} s_i I_{\langle M\rangle}(D_i)\right|\right|_{L^1_{\mathbb{R}}} \\
&= E\left(\sum_{i=1}^{n} s_i I_{\langle M\rangle}(D_i)\right) \\
&= \sum_{i=1}^{n} s_i E(I_{\langle M\rangle}(D_i)) \\
&= \sum_{i=1}^{n} s_i ||I_{\langle M\rangle}(D_i)||_{L^1_{\mathbb{R}}} \\
&= \sum_{i=1}^{n} s_i ||I_{|M|^2_E}(D_i)||_{L^1_{\mathbb{R}}} \\
&= \sum_{i=1}^{n} s_i E(I_{|M|^2_E}(D_i)) \\
&= \left|\left|\int S\ dI_{|M|^2_E}\right|\right|_{L^1_{\mathbb{R}}}.
\end{aligned} \tag{4.110}$$

The second equality in (4.110) follows from the fact that $\sum_{i=1}^{n} s_i I_{\langle M\rangle}(D_i)$ is positive P.a.s. (see our earlier comments in this remark; also note that $s_i \geq 0$ for $i = 1, \dots, n$). The fourth equality in (4.110) is similarly true. The fifth equality is true by (4.99) and (c) of this remark. The remainder of the steps in (4.110) follow the preceding pattern. For a positive predictable function H, using definition (2.3.29), we obtain

$$\begin{aligned}
(\tilde{I}_{\langle M\rangle})_{\mathbb{R},L^2_{\mathbb{R}}}(H) &= \sup\left|\left|\int S\ dI_{\langle M\rangle}\right|\right|_{L^1_{\mathbb{R}}} \\
&= \sup\left|\left|\int S\ dI_{|M|^2_E}\right|\right|_{L^1_{\mathbb{R}}} \\
&= (\tilde{I}_{|M|^2_E})_{\mathbb{R},L^1_{\mathbb{R}}}(H).
\end{aligned} \tag{4.111}$$

The sup in (4.111) is taken over all positive, simple, predictable function. The second equality in (4.111) is by (4.110). Hence we see that for any $H \geq 0$, predictable we

have

$$H \in \mathcal{F}_{\mathbb{R},L^1_{\mathbb{R}}}(I_{\langle M\rangle}) \text{ iff } H \in \mathcal{F}_{\mathbb{R},L^1_{\mathbb{R}}}(I_{|M|^2_E})$$

and such an H satisfies the equality in (4.111).

THEOREM (4.3.5). Let E be a Hilbert space and let M be an E-valued orthogonal martingale measure which satisfies the condition in (4.13). Let $H \in L^1_{\mathbb{R},L^2_E}(\mathcal{B}, M)$ (observe: H is real valued).

Then we have

$$(\tilde{I}_M)_{\mathbb{R},L^2_E}(H) = \left\|\int H \, dI_M\right\|_{L^2_E} = ||H||_{L^2_E(\mu_{\langle M\rangle})} \tag{1}$$

and

$$(\tilde{I}_M)_{\mathbb{R},L^2_E}(H) = \sqrt{(\tilde{I}_{\langle M\rangle})_{\mathbb{R},L^1_{\mathbb{R}}}(H^2)} = \sqrt{(\tilde{I}_{|M|^2_E})_{\mathbb{R},L^1_{\mathbb{R}}}(H^2)} \tag{2}$$

REMARK (4.3.6). The second equality in statement (1) of the theorem is the Ito isometry. This isometry $H \longrightarrow H \cdot M$ is from $L^1_{\mathbb{R},L^2_E}(\mathcal{B}, M)$ into $L^2_E\ (\mu_{\langle M\rangle})$.

Proof (of theorem (4.3.5)). We will prove the equalities in (1); statement (2) will follow as a by-product. Observe that for a cube in $\mathcal{R}$ of the form $[0_A] \times B, A \in \mathcal{F}_0$, $B \in \mathcal{B}(\mathbb{R})$ and for $r \in \mathbb{R}$ we have

$$\begin{aligned} ||I_M([0_A] \times B)r||^2_{L^2_E} &= ||1_A M_0(B) r||_{L^2_E} \\ &= E[1_A |M_0(B)|^2_E r^2] \\ &= E\left[\int 1_{[0_A]\times B} r^2 \, dI_{|M|^2_E}\right]. \end{aligned} \tag{4.112}$$

And for a cube in $\mathcal{R}$ of the form $(s,t] \times A \times B$ with $A \in \mathcal{F}_s$ and $B \in \mathcal{B}(\mathbb{R})$, we have

$$\begin{aligned} ||I_M((s,t] \times A \times B)r||^2_{L^2_E} &= ||1_A(M_t(B) - M_s(B))r||_{L^2_E} \\ &= E[1_A(M_t(B) - M_s(B))^2 r^2] \\ &= E[1_A(|M_t(B)|^2_E - |M_s(B)|^2_E) r^2] \\ &= E\left[\int_{(s,t]\times A\times B} r^2 \, dI_{|M|^2_E}\right]. \end{aligned} \tag{4.113}$$

The third equality in (4.113) is obtained in the same way as in (4.72). Now for a simple process H (see remark (3.2.4)) of the form

$$H = 1_{[0_A]} 1_{B_0} r_0 + \sum_{i=1}^{n} 1_{(s_i,t_i]} 1_{A_i} 1_{B_i} r_i \tag{4.114}$$

where $A_0 \in \mathcal{F}_0$, $A_i \in \mathcal{F}_{s_i}$ for $i = 1, \ldots, n$, $B_i \in \mathcal{B}(\mathbb{R})$ for $i = 0, \ldots, n$ and $r_i \in \mathbb{R}$ for $i = 0, \ldots, n$, and where the sets in the family $\{[0_A] \times B, (s_i, t_i] \times A_i \times B_i\}_{i=1}^n$ are mutually disjoint, we have

$$
\begin{aligned}
\left\|\int H \, dI_M\right\|_{L_E^2}^2 &= \|I_M([0_A] \times B) r_0\|_{L_E^2}^2 \\
&\quad + \sum_{i=1}^n \|I_M(s_i, t_i] \times A_i \times B_i) r_i\|_{L_E^2}^2 \\
&= E\left[\int 1_{[0_A] \times B} r_0^2 \, dI_{|M|_E^2}\right] \\
&\quad + \sum_{i=1}^n E\left[\int 1_{(s_i,t_i] \times A_i \times B_i} r_i^2 \, dI_{|M|_E^2}\right] \\
&= E\left[\int H^2 \, dI_{|M|_E^2}\right] \\
&= \int H^2 \, d\mu_{|M|_E^2} \\
&= \int H^2 \, d\mu_{\langle M \rangle} \\
&= E\left[\int H^2 \, dI_{\langle M \rangle}\right].
\end{aligned}
\tag{4.115}
$$

The first equality in (4.115) is an application of lemma (4.2.9). The fourth equality is true because $\mu_{|M|_E^2} \equiv E[I_{|M|_E^2}]$ on $\mathcal{R}$ and as H is $\mathcal{R}$-simple. The fifth equality is by (4.107) and the sixth is true since $\mu_{\langle M \rangle} = E[I_{\langle M \rangle}]$ on $\mathcal{R}$. Note that the Ito isometry $\|\int H \, dI_M\|_{L_E^2} = \|H\|_{L^2(\mu_{\langle M \rangle})}$ is evident from (4.115) for $\mathcal{R}$-simple functions H of the form in (4.114).

Since the family of measures $(I_M)_{\mathbb{R}, L_E^2}$ is uniformly σ-additive, we have by proposition (2.3.39) that

$$
\mathcal{F}_{\mathbb{R}, L_E^2}(S(\mathcal{R}), M) = \mathcal{F}_{\mathbb{R}, L_E^2}(\mathcal{B}, M). \tag{4.116}
$$

However,

$$
\mathcal{F}_{\mathbb{R}, L_E^2}(\mathcal{B}, M) = L^1_{\mathbb{R}, L_E^2}(\mathcal{B}, M). \tag{4.117}
$$

To prove the equality in (4.117) we need to check the conditions in definition (3.2.2). From part (b) of theorem (2.3.42), which uses the fact that $(I_M)_{\mathbb{R}, L_E^2}$-is uniformly

σ-additive, we know that for $H \in \mathcal{F}_{\mathbb{R},L^2_E}(\mathcal{B}, M)$ the set function $\int H\ dI_M : \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R}) \longrightarrow (L^2_E)^{**} = (L^2_E)$ is a σ-additive measure (note that $(L^2_E)^{**} = L^2_E$ as E is reflexive). In particular for any $t \geq 0$ fixed we have that $\int_{[0,t]\times(\cdot)} H\ dI_M : \mathcal{B}(\mathbb{R}) \longrightarrow L^2_E$ is a σ-additive measure. This takes care of condition (1). Now for each $t \geq 0$ and each $B \in \mathcal{B}(\mathbb{R})$ we know that $\int_{[0,t]\times B} H\ dI_M \in L^2_E = L^2_E(\Omega, \mathcal{F}, P)$. It follows from theorem (3.2.1) that $\int_{[0,t]\times B} H\ dI_M$ is $\mathcal{F}_t$-adapted. To complete the verification of condition (2) we need to show that the process $(\int_{[0,t]\times B} H\ dI_M)_{t\geq 0}$ has a cadlag modification for each $B \in \mathcal{B}(\mathbb{R})$ fixed. By the proof of theorem (3.6.1) (where we did not use the cadlag assumption to show that $(H \cdot M)$ is a martingale) we deduce that the process $(\int_{[0,t]\times B} H\ dI_M)$ is a martingale. It follows that $(\int_{[0,t]\times B} H\ dI_M)_{t\geq 0}$ has a cadlag modification (see part (3) of remark (3.6.2)). Thus definition (3.2.2) is satisfied for the H above, which means that $H \in L^1_{\mathbb{R},L^2_E}(\mathcal{B}, M)$. This proves (4.117). From (4.116) and (4.117) we deduce that the $\mathcal{R}$ simple functions are dense in $L^1_{\mathbb{R},L^2_E}(\mathcal{B}, M)$.

Now let $H \in L^1_{\mathbb{R},L^2_E}(\mathcal{B}, M)$ and let $(H^n)_{i=1}^\infty$ be a sequence of $\mathcal{R}$-simple functions which converge to H in $L^1_{\mathbb{R},L^2_E}(\mathcal{B}, M)$. Taking a subsequence we may assume that $H^n \longrightarrow H$, I_M a.s. Since $\mu_{\langle M\rangle} << I_M$ we deduce that $H^n \longrightarrow H$, $\mu_{\langle M\rangle}$ a.s. Using the isometry in (4.115), with H there replaced by $H^n - H^m$, we have

$$||H^n - H^m||_{L^2_{\mathbb{R}}(\mu_{\langle M\rangle})} = \left|\left|\int H^n - H^m\ dI_M\right|\right|_{L^2_E} \leq (\tilde{I}_M)_{\mathbb{R},L^2_E}(H^n - H^m) \longrightarrow 0.$$

Hence $(H^n)_{n=1}^\infty$ is cauchy in $L^2_{\mathbb{R}}(\mu_{\langle M\rangle})$. Since $L^2_{\mathbb{R}}(\mu_{\langle M\rangle})$ is complete and as $H^n \longrightarrow H, \mu_{\langle M\rangle}$ a.s. we deduce that (1) $H \in L^2_{\mathbb{R}}(\mu_{\langle M\rangle})$ and (2) $H^n \longrightarrow H$ in $L^2_{\mathbb{R}}(\mu_{\langle M\rangle})$. From the isometry it follows that

$$\left|\left|\int H\ dI_M\right|\right|_{L^2_E} = ||H||_{L^2_{\mathbb{R}}(\mu_{\langle M\rangle})}. \tag{4.118}$$

This gives the second equality in (1). The first equality in (1) is obtained as follows:

$$\begin{aligned}
(\tilde{I}_M)_{\mathbb{R},L^2_E}(H) &= \sup\left\|\int S\ dI_M\right\|_{L^2_E} \\
&= \sup ||S||_{L^2_{\mathbb{R}}(\mu_{\langle M\rangle})} \\
&= ||H||_{L^2_{\mathbb{R}}(\mu_{\langle M\rangle})} \\
&= \left\|\int H\ dI_M\right\|_{L^2_E}.
\end{aligned} \tag{4.119}$$

The sup in (4.119) is taken over all simple functions S such that $|S| \leq |H|$. The fourth equality in (4.119) is by (4.118). This proves statement (1) of the theorem. As for statement (2), let us compute $(\tilde{I}_{\langle M\rangle})_{\mathbb{R},L^1_{\mathbb{R}}}(H^2)$. We have

$$\begin{aligned}
(\tilde{I}_{\langle M\rangle})_{\mathbb{R},L^1_{\mathbb{R}}}(H^2) &= \sup\left\|\int S\ dI_{\langle M\rangle}\right\|_{L^1_{\mathbb{R}}} \\
&= \sup\int S\ d\mu_{\langle M\rangle} \\
&= ||H||^2_{L^2_{\mathbb{R}}(\mu_{\langle M\rangle})}
\end{aligned} \tag{4.120}$$

where the sup in (4.120) is over all predictable simple functions S such that $|S| \leq H^2$. The second equality in (4.120) is valid because (1) we may take S to be a (positive) simple function and (2) $I_{\langle M\rangle}(D) \geq 0$ a.s. for $D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$ - - see proposition (4.2.11). Combining (4.119) and (4.120) gives the first equality in (2). The second equality in (2) follows, at once, from (4.111). This completes the proof of the theorem. ∎

Our next, and last, theorem of this section says that, under the hypothesis of theorem (4.3.5), $(H \cdot M)$ is an orthogonal martingale measure that is 2-summable relative to $(\mathbb{R}, L^2_E)$.

THEOREM (4.3.7). Let E be a Hilbert space and let M be an E-valued orthogonal martingale measure which satisfies the condition in (4.13). Let $H \in L^1_{\mathbb{R},L^2_E}(\mathcal{B}, M)$. Then

(1) $(H \cdot M)$ is a martingale measure which is 2-summable relative to $(\mathbb{R}, L^2_E)$. and

(2) $(H \cdot M)$ is an orthogonal martingale measure.

PROOF. (1). That $(H \cdot M)$ is a 2-summable process measure follows from theorem (3.4.3) once we recall from the proof of theorem (4.3.5) that $\int_D H dI_M \in L^2_E$ for every $D \in \mathcal{P}_\infty \otimes \mathcal{B}(\mathbb{R})$. And from theorem (3.6.1) we know that $(H \cdot M)$ is a martingale. This proves (1).

(2). It remains for us to show that $(H \cdot M)$ is orthogonal. That is for $B, C \in \mathcal{B}(\mathbb{R})$ with $B \cap C = \emptyset$, we need to demonstrate that

$$\langle (H \cdot M)(B), (H \cdot M)(C) \rangle \equiv \langle H \cdot M \rangle (B, C) = 0. \tag{4.121}$$

Now, for any $A \in \mathcal{B}(\mathbb{R})$, $(H \cdot M)_t(A)$ is a square integrable martingale and hence we know that the difference

$$(H \cdot M)_t(B)(H \cdot M)_t(C) - \langle H \cdot M \rangle_t(B, C) \tag{4.122}$$

is a martingale. In order to show that $\langle H \cdot M \rangle (B, C) = 0$ it suffices to show that the process $((H \cdot M)_t(B)(H \cdot M)_t(C))_{t \geq 0}$ is a martingale. For then, by the unicity of the sharp bracket, we will obtain that $\langle H \cdot M \rangle_t (B, C) = 0$.

We consider first the case when H is a $\mathcal{R}$-simple process of the form

$$H = 1_{[0_A]} 1_{B_0} r_0 + \sum_{i=1}^{n} 1_{(s_i, t_i]} 1_{A_i} 1_{B_i} r_i \tag{4.123}$$

where $A \in \mathcal{F}_0$, $A_i \in \mathcal{F}_{s_i}$ for $i = 1, \ldots, n$, $B_i \in \mathcal{B}(\mathbb{R})$ and $r_i \in \mathbb{R}$ for $i = 0, \ldots, n$. In order to show that the process $((H \cdot M)_t(B)(H \cdot M)_t(C))_{t \geq 0}$ is a martingale it suffices by Dellacherie–Meyer [10] (chapter VI, theorem 13) to show that for every predictable stopping time T we have: (a) $(H \cdot M)_T(B)(H \cdot M)_T(C)$ is integrable and (b) $E[(H \cdot M)_T(B)(H \cdot M)_T(C)] = 0$. Computing $(H \cdot M)_t(B)$ and $(H \cdot M)_t(C)$ for the H in (4.123) we obtain

$$(H \cdot M)_t(B) = \sum_{i=1}^{n} 1_{A_i}(M_{t_i \wedge t}(B_i \cap B) - M_{s_i \wedge t}(B_i \cap B))$$

and (4.124)

$$(H \cdot M)_t(C) = \sum_{i=1}^{n} 1_{A_i}(M_{t_i \wedge t}(B_i \cap C) - M_{s_i \wedge t}(B_i \cap C)).$$

Observe from (4.124) that we may write $(H \cdot M)_t^{t_n}(B)$ for $(H \cdot M)_t(B)$ and similarly $(H \cdot M)_t^{t_n}(C)$ for $(H \cdot M)_t(C)$. Hence in order to confirm (a) and (b) it suffices to only consider those predictable stopping times bounded by t_n. We remark that in (4.124) we have used the fact that M is zero at zero. Now let T be a stopping time bounded by t_n and let $A \in \mathcal{B}(\mathbb{R})$. Then

$$\begin{aligned} (H \cdot M)_T(A) &= (H \cdot M)_T^{t_n}(A) &= (H \cdot M)_{T \wedge t_n}(A) \\ &= (H \cdot M)^T(A) &= ((1_{[[0,T]]}H) \cdot M)(A) = (H \cdot M^T)(A). \end{aligned} \tag{4.125}$$

Equalities four and five in (4.125) follow from part (2) of theorem (3.4.6). From (4.125) we have

$$\begin{aligned} \|(H \cdot M)_T(A)\|_{L_E^2} &= \|(H \cdot M^T)(A)\|_{L_E^2} &\leq (\tilde{I}_{M^T})_{\mathbb{R},L_E^2}(1_A H) \\ &= (\tilde{I}_M)_{\mathbb{R},L_E^2}(1_{[[0,T]]}1_A H) &\leq (\tilde{I}_M)_{\mathbb{R},L_E^2}(H) < \infty. \end{aligned} \tag{4.126}$$

The first inequality in (4.126) is by the same argument as in (3.58). The second equality is by theorem (3.4.5). The semivariation of H relative to $(\mathbb{R}, L_E^2)$ is finite as $H \in L^1_{\mathbb{R},L_E^2}(M)$. By Hölders' inequality we have

$$\|(H \cdot M)_T(B)(H \cdot M)_T(C)\|_{L_E^1} \leq \|(H \cdot M)_T(B)\|_{L_E^2}\|(H \cdot M)_T(C)\|_{L_E^2} < \infty.$$

This takes care of condition (a). We now establish condition (b). A glance at (4.124) indicates that in order to show (b) it suffices to prove that

$$\begin{aligned} E[1_{A_i}1_{A_j}(M_{t_i \wedge T}(B_i \cap B) - M_{s_i \wedge T}(B_i \cap B))(M_{t_j \wedge T}(B_j \cap C) \\ -M_{s_j \wedge T}(B_j \cap C))] = 0. \end{aligned} \tag{4.127}$$

We may write the left hand side of (4.127) as

$$E[I_{M^T}((s_i, t_i] \times A_i \times (B_i \cap B)) \; I_{M^T}((s_j, t_j] \times A_j \times (B_j \cap C))]. \tag{4.128}$$

Note that in (4.128) M^T is an orthogonal martingale measure (use proposition (3.3.1) and theorem (3.4.5)). The orthogonality of M^T follows by properties of the sharp bracket. That is,

$$\langle M^T(B), M^T(C)\rangle_t = \langle M(B), M(C)\rangle_t^T = 0,$$

(see [10], number (41.3), chapter VII). But the expression in (4.128) is zero by the proof of theorem (4.2.8) (see (4.77)). This shows that $E[(H\cdot M)_T(B)(H\cdot M)_T(C)] = 0$ for T a predictable stopping time and H a $\mathcal{R}$-simple function. As discussed earlier this proves that the process $((H \cdot M)_t(B)(H \cdot M)_t(C))_{t\geq 0}$ is a martingale and hence $\langle (H \cdot M)(B), (H \cdot M)(C)\rangle = 0$. We now attend to the general case. Let $H \in L^1_{\mathbb{R},L^2_E}(\mathcal{B}, M)$. By the proof of theorem (4.3.5) we know that the $\mathcal{R}$-simple functions are dense in $L^1_{\mathbb{R},L^2_E}(\mathcal{B}, M)$. Thus let $(H^n)_{n=1}^{\infty}$ be a sequence of $\mathcal{R}$-simple functions such that $H^n \longrightarrow H$ in $L^1_{\mathbb{R},L^2_E}(\mathcal{B}, M)$. We will now show that $(H \cdot M)$ satisfies conditions (a) and (b). Let T be any predictable stopping time and let $A \in \mathcal{B}(\mathbb{R})$. Then we have

$$\left\| \int_{[[0,T]]\times A} H^n \, dI_M - \int_{[[0,T]]\times A} H \, dI_M \right\|_{L^2_E} \leq (\tilde{I}_M)_{\mathbb{R},L^2_E}(H^n - H) \longrightarrow 0. \tag{4.129}$$

Since we have

$$\begin{aligned} \|(H \cdot M)_T(A)\|_{L^2_E} &\leq (\tilde{I}_M)_{\mathbb{R},L^2_E}(H^n - H) + \|(H^n \cdot M)_T(A)\|_{L^2_E} \\ &= (\tilde{I}_M)_{\mathbb{R},L^2_E}(H^n - H) + \|(H^n \cdot M^T)(A)\|_{L^2_E} \end{aligned}$$

it follows that $(H \cdot M)$ satisfies condition (a). To check (b) we observe from (4.129) that

$$(H^n \cdot M)_T(A) \longrightarrow (H \cdot M)_T(A) \text{ in } L^2_E$$

and hence it follows that for $B, C \in \mathcal{B}(\mathbb{R})$, disjoint, we have

$$(H^n \cdot M)_T(B)(H^n \cdot M)_T(C) \longrightarrow (H \cdot M)_T(B)(H.M)_T(C) \text{ in } L^1_E. \tag{4.130}$$

From (4.130) we also have

$$E[(H^n \cdot M)_T(B)(H^n \cdot M)_T(C)] \longrightarrow E[(H \cdot M)_T(B)(H \cdot M)_T(C)]. \tag{4.131}$$

Since the left side of (4.131) is zero for each n, we conclude that

$$E[(H \cdot M)_T(B)(H \cdot M)_T(C)] = 0.$$

Thus condition (b) is also satisfied by $(H\cdot M)$ and this means that $\langle H\cdot M\rangle(B, C) = 0$. The theorem is now proved. ∎

4.4 A Special Case of Martingale Measures with Nuclear Covariance

The concept of a martingale measure with nuclear covariance has been introduced in chapter 2 (definition (2.1.1)). In this section we prove that a canonical example of such a process measure is 2-summable. We present this example as a proposition.

PROPOSITION (4.4.1). Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \eta)$ be a finite measure space and suppose that $f \in L^2_{\mathbb{R}}(\mathbb{R}, \mathcal{B}(\mathbb{R}), \eta)$ with $f \geq 0$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Let $\{B_t, t \in [0,T], T < \infty\}$ be the standard (one-dimensional) Brownian motion and let $\{\mathcal{F}_t, t \in [0,T], T < \infty\}$ be the associated Brownian filtration. Consider the process measure $\{M_t(C), (\mathcal{F}_t),\ t \in [0,T], C \in \mathcal{B}(\mathbb{R})\}$ defined by

$$M_t(C)(w) = B_t(w) \int_C f \, d\eta. \tag{4.132}$$

Then,

(1) $\{M_t(C), (\mathcal{F}_t), t \in [0,T],\ C \in \mathcal{B}(\mathbb{R})\}$ is a martingale measure with nuclear covariance.

(2) I_M can be extended to a σ-additive measure on $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})$ with bounded semivariation relative to $(\mathbb{R}, L^2_{\mathbb{R}})$ ($\mathcal{P}_T$ is the predictable σ-algebra on $[0,T] \times \Omega$).

Proof. It is evident from (4.132) that M is a martingale measure. We need to check conditions (i) and (ii) in definition (2.1.1), part IV, to show that M has nuclear covariance. Condition (i) is obvious. As for condition (ii), let $(\phi_n)_{n=1}^{\infty}$ be a complete orthonormal system of real valued step functions, where for each n we have

$$\phi_n = \sum_{i=1}^{i_n} a_{n,i} I_{A_{n,i}};\ (A_{n,i})_{i=1}^{i_n} \text{ is a disjoint family in } \mathcal{B}(\mathbb{R}). \tag{4.133}$$

From definition (2.1.1) we know that $M_t(\phi_n)$ is defined by

$$M_t(\phi_n) = \sum a_{n,i} M_t(A_{n,i}).$$

Then we have

$$\begin{aligned}
E[(M_t(\phi_n))^2] &= E\left[\left(\sum_{i=1}^{i_n} a_{n,i}M_t(A_{n,i})\right)^2\right] \\
&= E\left[\left(\sum_{i=1}^{i_n} a_{n,i}B_t(w)\int_{A_{n,i}} f\ d\eta\right)^2\right] \\
&= \left(\sum_{i=1}^{i_n} a_{n,i}\int_{A_{n,i}} f\ d\eta\right)^2 \\
&= \left(\int \phi_n f\ d\eta\right)^2 \equiv \langle f, \phi_n\rangle^2,
\end{aligned}$$

where the last equality holds by Parseval's formula. This establishes condition (ii) in definition (2.1.1), part IV. Statement (1) of the proposition is now proved. For statement (2), we first show that I_M can be extended to a σ-additive measure on $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})$.

Let $\mathcal{R}_T$ denote the ring (of finite (disjoint) union of cubes of the form $(s,t]\times A\times B$, with $A \in \mathcal{F}_s$) which generates $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})$. Let $z \in (L^2_{\mathbb{R}})^* = L^2_{\mathbb{R}}$. Then for a cube $(s,t] \times A \times C$ with $A \in \mathcal{F}_s$, and $C \in \mathcal{B}(\mathbb{R})$ we have

$$\begin{aligned}
\langle I_M((s,t] \times A \times C), z\rangle &= \langle 1_A[M_t(C) - M_s(C)], z\rangle \\
&= \langle 1_A(B_t - B_s)\int_C f\ d\eta, z\rangle \\
&= \langle 1_A(B_t - B_s), z\rangle \int_C f\ d\eta \\
&= \langle I_B((s,t] \times A), z\rangle \textstyle\int_C f\ d\eta.
\end{aligned} \tag{4.134}$$

In (4.134) the stochastic measure I_B is, infact, a σ-additive measure over $\mathcal{P}_T$. This obtains from proposition 3.22 in Brooks and Dinculeanu [4], as B is a square integrable martingale. Hence $\langle I_B(\cdot), z\rangle : \mathcal{P}_T \longrightarrow \mathbb{R}$ is a σ-additive measure (A strongly σ-additive measure is, of course, weakly σ-additive). Let $\mu : \mathcal{B}(\mathbb{R}) \longrightarrow \mathbb{R}_+$ denote the positive, σ-additive measure $\mu(C) = \int_C f\ d\eta$ for $C \in \mathcal{B}(\mathbb{R})$. Then from (4.134) we have the following equality.

$$\langle I_M((s,t] \times A \times C), z\rangle = \langle I_B((s,t] \times A), z\rangle\mu(C). \tag{4.135}$$

Now let $\mathcal{R}'_T$ denote the ring of finite (disjoint) unions of rectangles (of the form $(s,t] \times A$, $A \in \mathcal{F}_s$ and $\{0\} \times A$, $A \in \mathcal{F}_0$) generating $\mathcal{P}_T$. One easily checks that the (bi) set function $I_M : \mathcal{R}'_T \times \mathcal{B}(\mathbb{R}) \longrightarrow L^2_{\mathbb{R}}$ is finitely additive in each component (in the sense of a bimeasure). Thus by lemma (1.3.3) (and remark (1.3.4)) we deduce that I_M is finitely additive on $\mathcal{R}'_T \times \mathcal{B}(\mathbb{R})$ and hence on $\mathcal{R}_T$.

For an arbitrary $D \in \mathcal{R}_T$, say $D = \dot{\bigcup}_{i=1}^n D_i$, $D_i = (s_i, t_i] \times A_i \times C_i$ we have

$$\begin{aligned} \langle I_M(D), z\rangle &= \left\langle \sum_{i=1}^n I_M(D_i), z \right\rangle \\ &= \sum_{i=1}^n \langle I_M(D_i), z\rangle \\ &= \sum_{i=1}^n \langle I_B((s_i, t_i] \times A_i), z\rangle \mu(C_i). \end{aligned} \tag{4.136}$$

In (4.136) the first equality is by the finite additivity of I_M on $\mathcal{R}_T$ and the third equality is by (4.135). The statement in (4.136) says that the scalar set function $\langle I_M(\cdot)\rangle, z\rangle$ is equal to the product measure $\langle I_B(\cdot), z\rangle\mu(\cdot)$ on $\mathcal{R}_T$. Since $\langle I_B(\cdot), z\rangle\mu(\cdot)$ is a positive (finite) measure on $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})$ it follows by proposition (4.2.7) that $\langle I_M, z\rangle$ can be extended to a σ-additive measure on $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})$. It then follows by the theorem of Pettis ([18], theorem 1, page 318) that I_M is a σ-additive measure on $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})$. It remains to show that I_M has bounded semivariation relative to $(\mathbb{R}, L^2_{\mathbb{R}})$ on $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})$. However, in view of proposition (2.3.17), it suffices to show that I_M has bounded semivariation relative to $(\mathbb{R}, L^2_{\mathbb{R}})$ on $\mathcal{R}_T$. We do this now.

Let $D \in \mathcal{R}_T$, $D = \dot{\bigcup}_{i=1}^n D_i$, as in (4.136). We may further write D as a union $\dot{\bigcup}_{i=1}^m D'_i$, $D'_i \in \mathcal{R}_T$, such that if $D'_i = (s_i, t_i] \times A_i \times C_i$ and $D_j = (s_j, t_j] \times A_j \times C_j$ where $A_i \in \mathcal{F}_{s_i}$, $A_j \in \mathcal{F}_{s_j}$, $i \neq j$, then $(s_i, t_i] \cap (s_j, t_j] = \emptyset$. Let $(r_i)_{i=1}^n$ be elements in

$\mathbb{R}$ with $|r_i| \le 1$. Then we have

$$\begin{aligned}
\left\|\sum_{i=1}^{m} I_M(D_i')r_i\right\|_{L^2_{\mathbb{R}}}^2 &= \sum_{i=1}^{m} E[I_B((s_i,t_i]\times A_i))^2 r_i^2](\mu(C_i))^2 \\
&+2\sum_{i<j} E[I_B((s_i,t_i]\times A_i)I_B((s_j,t_j]\times A_j)r_i r_j]\mu(C_i)\mu(C_j) \\
&= \sum_{i=1}^{m} E[1_{A_i}(B_{t_i}-B_{s_i})^2 r_i^2] \\
&+2\sum_{i<j} E[r_i, r_j 1_{A_i\cap A_j}(B_{t_i}-B_{s_i})(B_{t_j}-B_{s_j})]\mu(C_i)\mu(C_j) \\
&\le \sum_{i=1}^{m} E[(B_{t_i}-B_{s_i})^2](\mu(C_i))^2 \\
&+2\sum_{i<j} E[(B_{t_i}-B_{s_i})(B_{t_j}-B_{s_j})]\mu(C_i)\mu(C_j) \\
&= \sum_{i=1}^{m} |t_i - s_i|(\mu(C_i))^2 + 0 \\
&\le T(\mu(\mathbb{R}))^2 < \infty.
\end{aligned} \tag{4.137}$$

The last equality in (4.136) is obtained by the properties of Brownian motion [(1) $E((B_t - B_s)^2) = |t-s|$ and (2) $E((B_{t_1} - B_{s_1})(B_{t_2} - B_{s_2})) = 0$ (independent increments)]. Also observe that $\sum_{i=1}^{m} |t_i - s_i| \le T$ as the intervals $\{(s_i,t_i]\}_{i=1}^{m}$ are disjoint. We conclude from (4.137) that $(\tilde{I}_M)_{\mathbb{R},L^2_{\mathbb{R}}}(D) \le T(\mu(\mathbb{R}))^2$. Thus I_M has bounded semivariation relative to $(\mathbb{R}, L^2_{\mathbb{R}})$ over $\mathcal{R}_T$. The proof of the proposition is now complete.

■

CHAPTER 5
CONCLUSION

In this dissertation we introduced the notion of a process measure. We discussed the stochastic measure induced by a process measure and then studied some of its properties (chapter 2).

The collective effort in chapters 3 and 4 represents the core of this dissertation.

In chapter 3 we constructed the stochastic integral with respect to a p-summable process measure and then studied some of its properties. The main theme of this chapter is that the stochastic integral is itself a process measure. Infact, we saw that the properties of the integrator, X, were transferred to the stochastic integral $H \cdot X$. As an illustration, we proved that if X is a p-summable process measure that is also a martingale then $(H \cdot X)$ also possesses these features.

The main result of this dissertation, on the 2-summability of orthogonal martingale measures (theorem (4.2.8)),is presented in chapter 4. This, together with our Ito isometry theorem (theorem (4.3.5.)) and representation of the stochastic integral as an orthogonal martingale measure (theorem (4.3.7)), when the integrator has the same characteristic, gives a nice account of the L^2 stochastic integral theory.

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BIOGRAPHICAL SKETCH

I was born in Madras, India. As far as history is concerned (and, perhaps, it is not) the only thing that is remarkable about my academic career is that I was a terribly poor mathematics student until my 12th grade. My major, as an undergraduate, was finance–and hence I did not learn any mathematics beyond Calculus III and Linear Algebra during that period. My interest in mathematics was sparked in earnest only when I became a graduate student.

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